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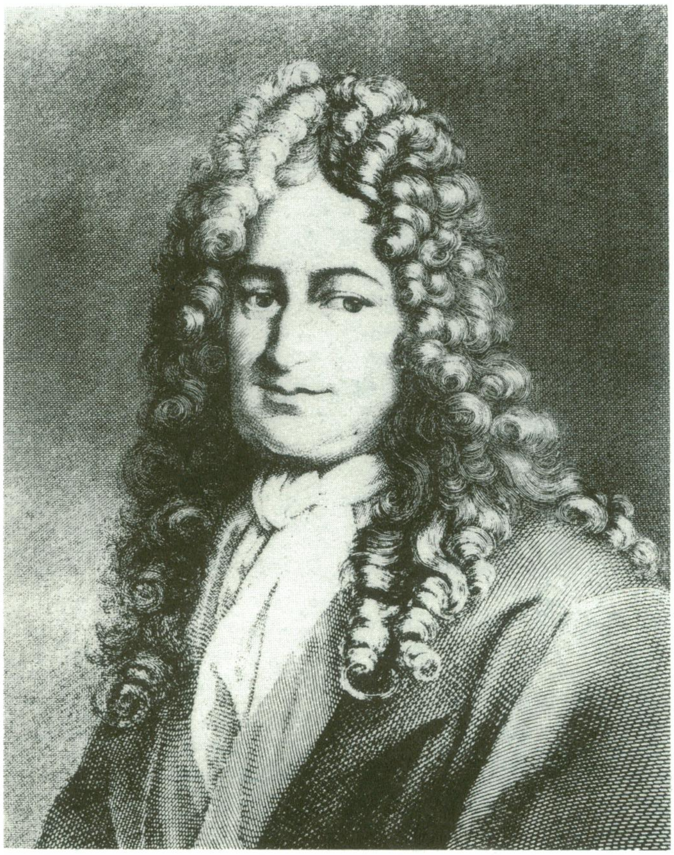

- Series Formula for π by Leibniz, Gregory and Nilakantha
- Why December 21 Is the Longest Day of the Year
- Karl Menger and Taxicab Geometry

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Cover: Gottfried Wilhelm, Baron von Leibniz (1646–1716)

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ARTICLES

The Discovery of the Series Formula for π by Leibniz, Gregory and Nilakantha

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1. Introduction

The formula for π mentioned in the title of this article is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots . \tag{1}$$

One simple and well-known modern proof goes as follows:

$$\begin{aligned} \arctan x &= \int_0^x \frac{1}{1+t^2}\,dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt. \end{aligned}$$

The last integral tends to zero if $|x| \leqslant 1$, for

$$\left| \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \right| \leqslant \left| \int_0^x t^{2n+2}\,dt \right| = \frac{|x|^{2n+3}}{2n+3} \to 0 \quad \text{as } n \to \infty.$$

Thus, arctan x has an infinite series representation for $|x| \leqslant 1$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots . \tag{2}$$

The series for $\pi/4$ is obtained by setting $x = 1$ in (2). The series (2) was obtained independently by Gottfried Wilhelm Leibniz (1646–1716), James Gregory (1638–1675) and an Indian mathematician of the fourteenth century or probably the fifteenth century whose identity is not definitely known. Usually ascribed to Nilakantha, the Indian proof of (2) appears to date from the mid-fifteenth century and was a consequence of an effort to rectify the circle. The details of the circumstances and ideas leading to the discovery of the series by Leibniz and Gregory are known. It is interesting to go into these details for several reasons. The infinite series began to play a role in mathematics only in the second half of the seventeenth century. Prior to that, particular cases of the infinite geometric series were the only ones to be used. The arctan series was obtained by Leibniz and Gregory early in their study of infinite series and, in fact, before the methods and algorithms of calculus were fully developed. The history of the arctan series is, therefore, important because it reveals early ideas on series and their relationship with quadrature or the process of finding the area under a curve. In the case of Leibniz, it is possible to see how he used and transformed older ideas on quadrature to develop his methods. Leibniz's work, in fact, was primarily concerned with quadrature; the $\pi/4$ series resulted (in 1673) when he applied his method to the circle. Gregory, by comparison, was interested in finding an infinite series representation of any given function and discovered the relationship between this and the successive derivatives of the given function. Gregory's discovery, made in 1671, is none other than the Taylor series; note that Taylor was not born until 1685. The ideas of calculus, such as integration by parts, change of variables, and higher derivatives, were not completely understood in the early 1670s. Some particular cases were known, usually garbed in geometric language. For example, the fundamental theorem of calculus was stated as a geometric theorem in a work of Gregory's written in 1668. Similar examples can also be seen in a book by Isaac Barrow, Newton's mentor, published in 1670. Of course, very soon after this transitional period, Leibniz began to create the techniques, algorithms and notations of calculus as they are now known. He had been preceded by Newton, at least as far as the techniques go, but Newton did not publish anything until considerably later. It is, therefore, possible to see how the work on arctan was at once dependent on earlier concepts and a transitional step toward later ideas.

Finally, although the proofs of (2) by Leibniz, Gregory and Nilakantha are very different in approach and motivation, they all bear a relation to the modern proof given above.

2. Gottfried Wilhelm Leibniz (1646–1716)

Leibniz's mathematical background[1] at the time he found the $\pi/4$ formula can be quickly described. He had earned a doctor's degree in law in February 1667, but had studied mathematics on his own. In 1672, he was a mere amateur in mathematics. That year, he visited Paris and met Christiaan Huygens (1629–1695), the foremost physicist and mathematician in continental Europe. Leibniz told the story of this meeting in a 1679 letter to the mathematician Tschirnhaus, "at that time... I did not know the correct definition of the center of gravity. For, when by chance I spoke of it to Huygens, I let him know that I thought that a straight line drawn through the center of gravity always cut a figure into two equal parts,... Huygens laughed when he heard this, and told me that nothing was further from the truth. So I, excited by this stimulus, began to apply myself to the study of the more intricate geometry, although as a matter of fact I had not at that time really studied the *Elements* [Euclid]... Huygens, who thought me a better geometer than I was, gave me to read the letters of Pascal, published under the name of Dettonville; and from these I gathered the method of indivisibles and centers of gravity, that is to say the well-known methods of Cavalieri and Guldinus."[2]

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[1]For further information about Leibniz's mathematical development, the reader may consult: J. E. Hofmann, *Leibniz in Paris 1672–1676* (Cambridge: The Cambridge University Press, 1974) and its review by A. Weil, *Collected Papers* Vol. 3 (New York: Springer-Verlag, 1979). An English translation of Leibniz's own account, *Historia et origo calculi differentialis*, can be found in J. M. Child, *The Early Mathematical Manuscripts of Leibniz* (Chicago: Open Court, 1920). An easily available synopsis of Leibniz's work in calculus is given in C. H. Edwards, Jr., *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979).

[2]*The Early Mathematical Manuscripts*, p. 215.

Bonaventura Cavalieri (1598–1647) published his *Geometria Indivisibilibus* in 1635. This book was very influential in the development of calculus. Cavalieri's work indicated that

The study of Pascal played an important role in Leibniz's development as a mathematician. It was from Pascal that he learned the ideas of the "characteristic triangle" and "transmutation." In order to understand the concept of transmutation, suppose A and B are two areas (or volumes) which have been divided up into indivisibles usually taken to be infinitesimal rectangles (or prisms). If there is a one-to-one correspondence between the indivisibles of A and B and if these indivisibles have equal areas (or volumes), then B is said to be obtained from A by transmutation and it follows that A and B have equal areas (or volumes). Pascal had also considered infinitesimal triangles and shown their use in finding, among other things, the area of the surface of a sphere. Leibniz was struck by the idea of an infinitesimal triangle and its possibilities. He was able to derive an interesting transmutation formula, which he then applied to the quadrature of a circle and thereby discovered the series for π. To obtain the transmutation formula, consider two neighboring points $P(x, y)$, and $Q(x+dx, y+dy)$ on a curve $y=f(x)$. First Leibniz shows that area $(\Delta OPQ) = (1/2)$ area (rectangle $(ABCD)$). See FIGURE 1. Here PT is tangent to $y=f(x)$ at P and OS is perpendicular to PT. Let p denote the length of OS and z that of $AC = BD =$ ordinate of T.

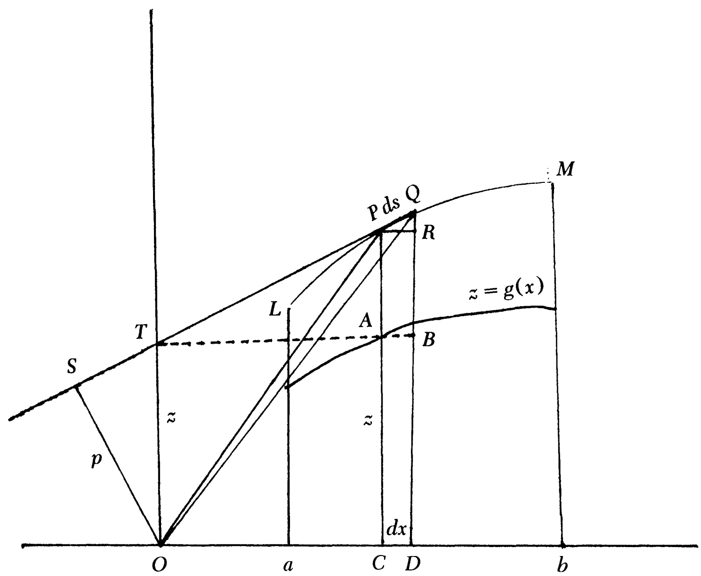

FIGURE 1

$$\int_0^a x^n\,dx = \frac{a^{n+1}}{n+1},$$

when n is a positive integer.

Blaise Pascal (1623–1662) made important and fundamental contributions to projective geometry, probability theory and the development of calculus. The work to which Leibniz refers was published in 1658 and contains the first statement and proof of

$$\int_{\theta_0}^{\theta} \sin\phi\,d\phi = \cos\theta_0 - \cos\theta.$$

This proof is presented in D. J. Struik's *A Source Book in Mathematics 1200–1800* (Cambridge: Harvard University Press, 1969), p. 239.

Paul Guldin (1577–1643), a Swiss mathematician of considerable note, contributed to the development of calculus, and his methods were generally more rigorous than those of Cavalieri.

Since ΔOST is similar to the characteristic ΔPQR,

$$\frac{dx}{p} = \frac{ds}{z},$$

where ds is the length of PQ. Thus,

$$\text{area}\,(OPQ) = \frac{1}{2}pds = \frac{1}{2}zdx. \tag{3}$$

Now, observe that for each point P on $y = f(x)$ there is a corresponding point A. Thus, as P moves from L to M, the points A form a curve, say $Z = g(x)$. If sector OLM denotes the closed region formed by $y = f(x)$ and the straight lines OL and OM, then (3) implies that

$$\text{area (sector } OLM) = \frac{1}{2}\int_a^b g(x)\,dx. \tag{4}$$

This is the transmutation formula of Leibniz. From (4), it follows that the area under $y = f(x)$ is

$$\begin{aligned}\int_a^b y\,dx &= \frac{b}{2}f(b) - \frac{a}{2}f(a) + \text{area (sector } OLM)\\ &= \frac{1}{2}\left([xy]_a^b + \int_a^b z\,dx\right).\end{aligned} \tag{5}$$

This is none other than a particular case of the formula for integration by parts. For it is easily seen from FIGURE 1 that

$$z = y - x\frac{dy}{dx}. \tag{6}$$

Substituting this value of z in (5), it follows that

$$\int_a^b y\,dx = [xy]_a^b - \int_{f(a)}^{f(b)} x\,dy,$$

which is what one gets on integration by parts.

Now consider a circle of radius 1 and center $(1,0)$. Its equation is $y^2 = 2x - x^2$. In this case, (6) implies that

$$z = \sqrt{2x - x^2} - \frac{x(1-x)}{\sqrt{2x-x^2}} = \sqrt{\frac{x}{2-x}} = \frac{x}{y}, \tag{7}$$

so that

$$x = \frac{2z^2}{1+z^2}. \tag{8}$$

In FIGURE 2, let $A\hat{O}B = 2\theta$. Then the area of the sector $AOB = \theta$ and

$$\theta = \text{area}\,(\triangle\,AOB) + \text{area (region between arc } AB \text{ and line } AB). \tag{9}$$

By the transmutation formula (4), the second area is $\frac{1}{2}\int_0^x z\,dt$ where z is given by (7). Now, from FIGURE 3 below it is seen that

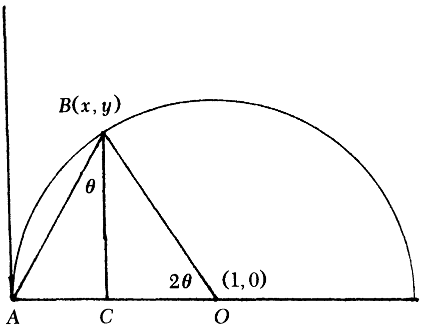

FIGURE 2

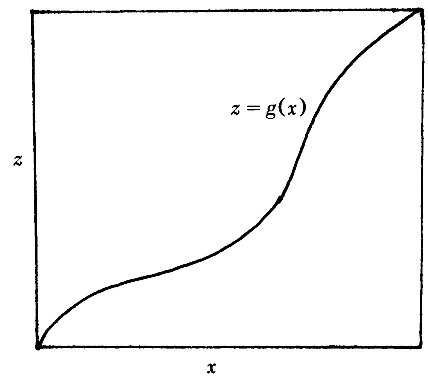

FIGURE 3

$$\frac{1}{2}\int_0^x z\,dt = \frac{1}{2}\left(xz - \int_0^z x\,du\right). \tag{10}$$

Using (8) and (10), it is now possible to rewrite (9) as

$$\begin{aligned}\theta &= \tfrac{1}{2}y + \tfrac{1}{2}xz - \int_0^z \frac{t^2}{1+t^2}\,dt \\ &= \tfrac{1}{2}[z(2-x) + xz] - \int_0^z \frac{t^2}{1+t^2}\,dt \quad (\text{since } y = z(2-x)) \\ &= z - \int_0^z \frac{t^2}{1+t^2}\,dt.\end{aligned}$$

At this point, Leibniz was able to use a technique employed by Nicolaus Mercator (1620–1687). The latter had considered the problem of the quadrature of the hyperbola $y(1+x) = 1$. Since it was already known that

$$\int_0^a x^n\,dx = \frac{a^{n+1}}{n+1},$$

he solved the problem by expanding $1/(1+x)$ as an infinite series and integrating term by term. He simultaneously had the expansion for $\log(1+x)$. Mercator published this result in 1668, though he probably had obtained it a few years earlier. A year later, John Wallis (1616–1703) determined the values of x for which the series is valid. Thus, Leibniz found that

$$\theta = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots. \tag{11}$$

In FIGURE 2, $A\hat{B}C = \theta$ and $z = x/y = \tan\theta$. Therefore, (11) is the series for arctan z.

Of course, Leibniz did not invent the notation for the integral and differential used above until 1675, and his description of the procedures is geometrical but the ideas are the same.

The discovery of the infinite series for π was Leibniz's first great achievement. He communicated his result to Huygens, who congratulated him, saying that this remarkable property of the circle will be celebrated among mathematicians forever. Even Isaac Newton (1642–1727) praised Leibniz's discovery. In a letter of October 24, 1676, to Henry Oldenburg, secretary of the Royal Society of London, he writes, "Leibniz's method for obtaining convergent series is certainly very elegant, and it would have sufficiently revealed the genius of its author, even if he had written nothing else."[3] Of course, for Leibniz this was only a first step to greater things as he himself says in his "Historia et origo calculi differentialis."

3. James Gregory (1638–1675)

Leibniz had been anticipated in the discovery of the series for arctan by the Scottish mathematician, James Gregory, though the latter did not note the particular case for $\pi/4$.[4] Since Gregory did not publish most of his work on infinite series and also because he died early and worked in isolation during the last seven years of his life, his work did not have the influence it deserved. Gregory's early scientific interest was in optics about which he wrote a masterly book at the age of twenty-four. His book, the *Optica Promota*, contains the earliest description of a reflecting telescope. It was in the hope, which ultimately remained unfulfilled, of constructing such an instrument that he travelled to London in 1663 and made the acquaintance of John Collins (1624–1683), an accountant and amateur mathematician. This friendship with Collins was to prove very important for Gregory when the latter was working alone at St. Andrews University in Scotland. Collins kept him abreast of the work of the English mathematicians such as Isaac Newton, John Pell (1611–1685) and others with whom Collins was in contact.[5]

Gregory spent the years 1664–1668 in Italy and came under the influence of the Italian school of geometry founded by Cavalieri. It was from Stefano degli Angeli (1623–1697), a student of Cavalieri, that Gregory learned about the work of Pierre de Fermat (1601–1665), Cavalieri, Evangelista Torricelli (1608–1647) and others. While in Italy, he wrote two books: *Vera Circuli et Hyperbolae Quadratura* in 1667, and *Geometriae Pars Universalis* in 1668. The first book contains some highly original ideas. Gregory attempted to show that the area of a general sector of an ellipse, circle or hyperbola could not be expressed in terms of the areas of the inscribed and circumscribed triangle and quadrilateral using arithmetical operations and root extraction. The attempt failed but Gregory introduced a number of important ideas such as convergence and algebraic and transcendental functions. The second book contains the first published statement and proof of the fundamental theorem of calculus in geometrical form. It is known that Newton had discovered this result not later than 1666, although he did not make it public until later.

Gregory returned to London in the summer of 1668; Collins then informed him of the latest discoveries of mathematicians working in England, including Mercator's recently published proof of

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots.$$

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[3]See H. W. Turnbull (ed.), *The Correspondence of Isaac Newton* (Cambridge: The University Press, 1960), Vol. 2, p. 130.

[4]Peter Beckmann has persuasively argued that Gregory must have known the series for $\pi/4$ as well. See Beckmann's *A History of Pi* (Boulder, Colorado: The Golem Press, 1977), p. 133.

[5]The reader might find it of interest to consult: H. W. Turnbull (ed.), *James Gregory Tercentenary Memorial Volume* (London: G. Bell, 1939). This volume contains Gregory's scientific correspondence with John Collins and a discussion of the former's life and work.

Meditation on these discoveries led Gregory to publish his next book, *Exercitationes Geometricae*, in the winter of 1668. This is a sequel to the *Pars Universalis* and is mainly about the logarithmic function and its applications. It contains, for example, the first evaluations of the indefinite integrals of $\sec x$ and $\tan x$.[6] The results are stated in geometric form.

In the autumn of 1668, Gregory was appointed to the chair in St. Andrews and he took up his duties in the winter of 1668/1669. He began regular correspondence with Collins soon after this, communicating to him his latest mathematical discoveries and requesting Collins to keep him informed of the latest activities of the English mathematicians. Thus, in a letter of March 24, 1670, Collins writes, "Mr. Newtone of Cambridge sent the following series for finding the Area of a Zone of a Circle to Mr. Dary, to compare with the said Dary's approaches, putting R the radius and B the parallel distance of a Chord from the Diameter the Area of the space or Zone between them is =

$$2RB - \frac{B^3}{3R} - \frac{B^5}{20R^3} - \frac{B^7}{56R^5} - \frac{5B^9}{576R^7}."[7]$$

This is all Collins writes about the series but it is, in fact, the value of the integral $2\int_0^B (R^2 - x^2)^{1/2}\,dx$ after expanding by the binomial theorem and term by term integration. Newton had discovered the binomial expansion for fractional exponents in the winter of 1664/1665, but it was first made public in the aforementioned letter of 1676 to Oldenburg.

There is evidence that Gregory had rediscovered the binomial theorem by 1668.[8] However, it should be noted that the expansion for $(1-x)^{1/2}$ does not necessarily

[6]A proof of the formula

$$\int_0^\theta \sec\phi\, d\phi = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

was of considerable significance and interest to mathematicians in the 1660's due to its connection with a problem in navigation. Gerhard Mercator (1512–1594) published his engraved "Great World Map" in 1569. The construction of the map employed the famous Mercator projection. Edward Wright, a Cambridge professor of mathematics, noted that the ordinate on the Mercator map corresponding to a latitude of θ^0 on the globe is given by $c\int_0^\theta \sec\phi\, d\phi$, where c is suitably chosen according to the size of the map. In 1599, Wright published this result in his *Certaine Errors in Navigation Corrected*, which also contained a table of latitudes computed by the continued addition of the secants of $1', 2', 3'$, etc. This approximation to $\int_0^\theta \sec\phi\, d\phi$ was sufficiently exact for the mariner's use. In the early 1640's, Henry Bond observed that the values in Wright's table could be obtained by taking the logarithm of $\tan(\pi/4 + \theta/2)$. This observation was published in 1645 in Richard Norwood's *Epitome of Navigation*. A theoretical proof of this observation was very desirable and Nicolaus Mercator had offered a sum of money for its demonstration in 1666. John Collins, who had himself written a book on navigation, drew Gregory's attention to this problem and, as we noted, Gregory supplied a proof. For more details, one may consult the following two articles by F. Cajori: "On an Integration ante-dating the Integral Calculus," *Bibliotheca Mathematica* Vol. 14 (1913/14), pp. 312–19, and "Algebra In Napier's Day and Alleged Prior Invention of Logarithms," in C. G. Knott (ed.), *Napier Memorial Volume* (London: Longmans, Green & Co., 1915), pp. 93–106. More recently, J. Lohne has established that Thomas Harriot (1560–1621) had evaluated the integral $\int_0^\theta \sec\phi\, d\phi$ in 1594 by a stereographic projection of a spherical loxodrome from the south pole into a logarithmic spiral. This work was unpublished and remained unknown until Lohne brought it to light. See J. A. Lohne, "Thomas Harriot als Mathematiker," *Centaurus*, Vol. 11, 1965–66, pp. 19–45. Thus it happened that, although $\int \sec\theta\, d\theta$ is a relatively difficult trigonometric integral, it was the first one to be discovered.

[7]*James Gregory*, p. 89.

[8]See *The Correspondence of Isaac Newton*, Vol. 1, p. 52, note 1.

imply a knowledge of the binomial theorem. Newton himself had proved the expansion by applying the well-known method for finding square roots of numbers to the algebraic expression $1-x$. Moreover, it appears that the expansion of $(1-x)^{1/2}$ was discovered by Henry Briggs (1556–1630) in the 1620's, while he was constructing the log tables.[9] But there is no indication that Gregory or Newton knew of this. In any case, for reasons unknown, Gregory was unable to make anything of the series—as evidenced by his reply of April 20, "I cannot understand the series you sent me of the circle, if this be the original, I take it to be no series."[10] However, by September 5, 1670, he had discovered the general interpolation formula, now called the Gregory-Newton interpolation formula, and had made from it a number of remarkable deductions. He now knew how "to find the sinus having the arc and to find the number having the logarithm." The latter result is precisely the binomial expansion for arbitrary exponents. For, if we take x as the logarithm of y to the base $1+d$, then $y=(1+d)^x$ and Gregory gives the solution as

$$(1+d)^x = 1 + xd + \frac{x(x-1)}{1\cdot 2}d^2 + \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}d^3 + \cdots .$$ [11]

It is possible that Newton's series in Collins' letter had set Gregory off on the course of these discoveries, but he did not even at this point see that he could deduce Newton's result. Soon after, he did observe this and wrote on December 19, 1670, "I admire much my own dullness, that in such a considerable time I had not taken notice of this."[12] All this time, he was very eager to learn about Newton's results on series and particularly the methods he had used. Finally on December 24, 1670, Collins sent him Newton's series for $\sin x$, $\cos x$, $\sin^{-1}x$ and $x\cot x$, adding that Newton had a universal method which could be applied to any function. Gregory then made a concentrated effort to discover a general method for himself. He succeeded. In a famous letter of February 15, 1671 to Collins he writes, "As for Mr. Newton's universal method, I imagine I have some knowledge of it, both as to geometrick and mechanick curves, however I thank you for the series ye sent me and send you these following in requital."[13] Gregory then gives the series for $\arctan x$, $\tan x$, $\sec x$, $\log\sec x$, $\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$, $\operatorname{arcsec}(\sqrt{2}\,e^x)$, and $2\arctan\tanh x/2$. However, what he had found was not Newton's method but rather the Taylor expansion more than forty years before Brook Taylor (1685–1731). Newton's method consisted of reversion of series, expansion by the binomial theorem, long division by series and term by term integration.[14] Thinking that he had rediscovered Newton's method, Gregory did not

[9] See D. T. Whiteside, "Henry Briggs: The Binomial theorem Anticipated," *The Mathematical Gazette*, Vol. 15, (1962), p. 9. Whiteside shows how the expansion of $(1+x)^{1/2}$ arose out of Brigg's work on logarithms.

[10] *James Gregory*, p. 92.

[11] In their review of the Gregory Memorial Volume, M. Dehn and E. Hellinger explain how the binomial expansion comes out of the interpolation formula. See *The American Mathematical Monthly*, Vol. 50, (1943), p. 149.

[12] *James Gregory*, p. 148.

[13] *Ibid.*, p. 170.

[14] It should be mentioned that Newton himself discovered the Taylor series around 1691. See D. T. Whiteside (ed.), *The Mathematical Papers of Isaac Newton*, Vol. VII (Cambridge: The Cambridge University Press, 1976), p. 19. In fact, Taylor was anticipated by at least five mathematicians. However, the

publish his results. It is only from notes that he made on the back of a letter from Gedeon Shaw, an Edinburgh stationer, dated January 29, 1671, that it is possible to conclude that Gregory had the idea of the Taylor series. These notes contain the successive derivatives of $\tan x$, $\sec x$, and the other functions whose expansions he sent to Collins. The following extract from the notes gives the successive derivatives of $\tan\theta$; here m is successively $y, \frac{dy}{d\theta}, \frac{d^2y}{d\theta^2}$, etc., and $q = r\tan\theta$. Gregory writes[15]:

1st	2nd	3rd	4th
$m = q$	$m = r + \frac{q^2}{r}$	$m = 2q + \frac{2q^3}{r^2}$	$m = 2r + \frac{8q^2}{r} + \frac{6q^4}{r^3}$

5th

$$m = 16q + \frac{40q^3}{r^2} + \frac{24q^5}{r^4}$$

6th

$$m = 16r + \frac{136q^2}{r} + \frac{240q^4}{r^3} + \frac{120q^6}{r^5}$$

7th

$$m = 272q + 987\frac{q^3}{r^2} + 1680\frac{q^5}{r^4} + 720\frac{q^7}{r^6}$$

8th

$$m = 272r + 3233\frac{q^2}{r} + 11361\frac{q^4}{r^3} + 13440\frac{q^6}{r^5} + 5040\frac{q^8}{r^7}.$$

It is clear from the form in which the successive derivatives are written that each one is formed by multiplying the derivative with respect to q of the preceding term by $r + \frac{q^2}{r}$. Now writing $a = r\theta$, Gregory gives the series in the letter to Collins as follows:

$$r\tan\theta = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{3233a^9}{181440r^8} + \cdots .$$

The reasons for supposing that these notes were written not much before he wrote to Collins and were used to construct the series are (i) the date of Gedeon Shaw's letter and (ii) Gregory's error in computing the coefficient of $\frac{q^3}{r^2}$ in the 7th m, which should be 1232 instead of 987 and which, in turn, leads to the error in the 8th m, where the coefficient of $\frac{q^2}{r}$ should be 3968 instead of 3233. This error is then repeated in the series showing the origin of the series. Moreover, in the early parts of the notes, Gregory is unsure about how he should write the successive derivatives. For example, he attempts to write the derivative of $\sec\theta$ as a function of $\sec\theta$ but then abandons the idea. He comes back to it later and sees that it is easier to work with m^2 instead of m since the m^2's can be expressed as polynomials in $\tan\theta$. This is, of course, sufficient to give him the series for $\sec\theta$. The series for $\log\sec\theta$ and $\log\tan(\pi/4 + \theta)$ he then obtains by term by term integration of the series for $\tan\theta$ and $\sec\theta$

Taylor series is not unjustly named after Brook Taylor who published it in 1715. He saw the importance of the result and derived several interesting consequences. For a discussion of these matters see: L. Feigenbaum, "Brook Taylor and the Method of Increments," *Archive for History of Exact Sciences*, Vol. 34, (1985), pp. 1–140.

[15] *James Gregory*, p. 352.

respectively. Naturally, the 3233 error is repeated. He must have obtained the series for arc tan x from the 2nd m which can be written as

$$\frac{da}{dq} = \frac{r^2}{r^2 + q^2} = 1 - \frac{q^2}{r^2} + \frac{q^4}{r^4} - \cdots .$$

The arctan series follows after term by term integration. Clearly, Gregory had made great progress in the study of infinite series and the calculus and, had he lived longer and published his work, he might have been classed with Newton and Leibniz as a co-discoverer of the calculus. Unfortunately, he was struck by a sudden illness, accompanied with blindness, as he was showing some students the satellites of Jupiter. He did not recover and died soon after in October, 1675, at the age of thirty-seven.

4. Kerala Gargya Nilakantha (c.1450–c.1550)

Another independent discovery of the series for arctan x and other trigonometric functions was made by mathematicians in South India during the fifteenth century. The series are given in Sanskrit verse in a book by Nilakantha called *Tantrasangraha* and a commentary on this work called *Tantrasangraha-vakhya* of unknown authorship. The theorems are stated without proof but a proof of the arctan, cosine and sine series can be found in a later work called *Yuktibhasa*. This was written in Malayalam, the language spoken in Kerala, the southwest coast of India, by Jyesthadeva (c.1500–c.1610) and is also a commentary on the *Tantrasangraha*. These works were first brought to the notice of the western world by an Englishman named C. M. Whish in 1835. Unfortunately, his paper on the subject had almost no impact and went unnoticed for almost a century when C. Rajagopal[16] and his associates began publishing their findings from a study of these manuscripts. The contributions of medieval Indian mathematicians are now beginning to be recognized and discussed by authorities in the field of the history of mathematics.[17]

It appears from the astronomical data contained in the *Tantrasangraha* that it was composed around the year 1500. The *Yuktibhasa* was written about a century later. It is not completely clear who the discoverer of these series was. In the *Aryabhatiya-bhasya*, a work on astronomy, Nilakantha attributes the series for sine to Madhava. This mathematician lived between the years 1340–1425. It is not known whether

[16]Rajagopal's work may be found in the following papers: (with M. S. Rangachari) "On an Untapped Source of Medieval Keralese Mathematics," *Archive for History of Exact Sciences*, Vol. 18, (1977), pp. 89–102; "On Medieval Kerala Mathematics," *Archive for History of Exact Sciences*, Vol. 35, (1986), pp. 91–99. These papers give the Sanskrit verses of the *Tantrasangrahavakhya* which describe the series for the arctan, sine and cosine. An English translation and commentary is also provided. A commentary on the proof of arctan series given in the *Yuktibhasa* is available in the two papers: "A Neglected Chapter of Hindu Mathematics," *Scripta Magnematica*, Vol. 15, (1949), pp. 201–209; "On the Hindu Proof of Gregory's Series," *Ibid.*, Vol. 17, (1951), pp. 65–74. A commentary on the *Yuktibhasa's* proof of the sine and cosine series is contained in C. Rajagopal and A. Venkataraman, "The sine and cosine power series in Hindu mathematics," *Journal of the Royal Asiatic Society of Bengal*, *Science*, Vol. 15, (1949), pp. 1–13.

[17]See J. E. Hofmann, "Über eine alt indische Berechnung von π und ihre allgemeine Bedeutung," *Mathematische-Physikalische Semester Berichte*, Bd. 3, H. 3/4, Hamburg (1953). See also D. T. Whiteside, "Patterns of Mathematical Thought in the later Seventeenth Century," *Archive for History of Exact Sciences*, Vol. 1, (1960–1962), pp. 179–388. For a discussion of medieval Indian mathematicians and the *Tantrasangraha* in particular, one might consult: A. P. Jushkevich, *Geschichte der Mathematik in Mittelalter* (German translation Leipzig, 1964, of the Russian original, Moscow, 1961).

Madhava found the other series as well or whether they are somewhat later discoveries.

Little is known about these mathematicians. Madhava lived near Cochin in the very southern part of India (Kerala) and some of his astronomical work still survives. Nilakantha was a versatile genius who wrote not only on astronomy and mathematics but also on philosophy and grammar. His erudite expositions on the latter subjects were well known and studied until recently. He attracted several gifted students, including Tuncath Ramanujan Ezuthassan, an early and important figure in Kerala literature. About Jyesthadeva, nothing is known except that he was a Brahmin of the house of Parakroda.

In the *Tantrasangraha-vakhya*, the series for arctan, sine and cosine are given in verse which, when converted to mathematical symbols may be written as

$$r \arctan\frac{y}{x} = \frac{1}{1}\cdot\frac{ry}{x} - \frac{1}{3}\cdot\frac{ry^3}{x^3} + \frac{1}{5}\cdot\frac{ry^5}{x^5} - \cdots, \text{ where } \frac{y}{x} \leqslant 1,$$

$$y = s - s\cdot\frac{s^2}{(2^2+2)r^2} + s\cdot\frac{s^2}{(2^2+2)r^2}\cdot\frac{s^2}{(4^2+4)r^2} - \cdots \quad \text{(sine)}$$

$$r - x = r\cdot\frac{s^2}{(2^2-2)r^2} - r\cdot\frac{s^2}{(2^2-2)r^2}\cdot\frac{s^2}{(4^2-4)r^2} + \cdots \quad \text{(cosine)}.$$

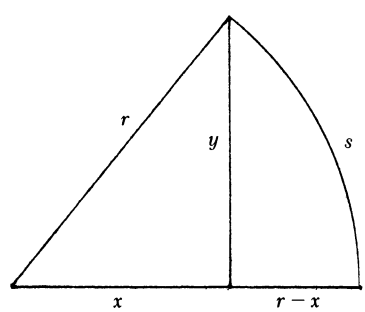

FIGURE 4

There are also some special features in the *Tantrasangraha's* treatment of the series for $\pi/4$ which were not considered by Leibniz and Gregory. Nilakantha states some rational approximations for the error incurred on taking only the first n terms of the series. The expression for the approximation is then used to transform the series for $\pi/4$ into one which converges more rapidly. The errors are given as follows:

$$\frac{\pi}{4} \simeq 1 - \frac{1}{3} + \frac{1}{5} - \cdots \mp \frac{1}{n} \pm f_i(n+1) \qquad i = 1,2,3, \tag{12}$$

where

$$f_1(n) = \frac{1}{2n},\ f_2(n) = \frac{n/2}{n^2+1} \text{ and } f_3(n) = \frac{(n/2)^2+1}{(n^2+5)n/2}.$$

The transformed series are as follows:

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \cdots \tag{13}$$

and

$$\frac{\pi}{4} = \frac{4}{1^5 + 4 \cdot 1} - \frac{4}{3^5 + 4 \cdot 3} + \frac{4}{5^5 + 4 \cdot 5} - \cdots .$$

Leibniz's proof of the formula for $\pi/4$ was found by the quadrature of a circle. The proof in Jyesthadeva's book is by a direct rectification of an arc of a circle. In the diagram given below, the arc AC is a quarter circle of radius one with center O and $OABC$ is a square. The side AB is divided into n equal parts of length δ so that $n\delta = 1$, $P_{r-1}P_r = \delta$. EF and $P_{r-1}D$ are perpendicular to OP_r. Now, the triangles OEF and $OP_{r-1}D$ are similar, which gives

$$\frac{EF}{OE} = \frac{P_{r-1}D}{OP_{r-1}}, \quad \text{that is, } EF = \frac{P_{r-1}D}{OP_{r-1}}.$$

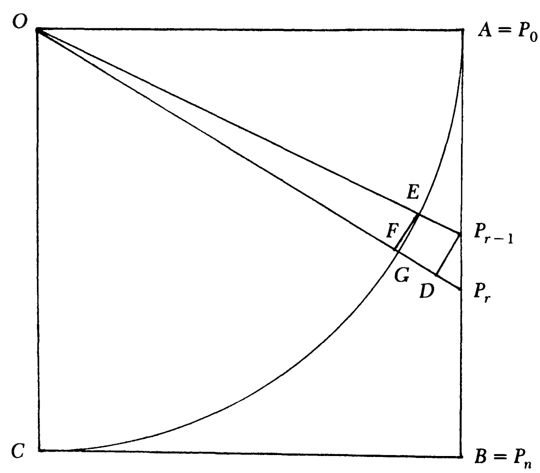

FIGURE 5

The similarity of the Δs $P_{r-1}P_rD$ and OAP_r gives

$$\frac{P_{r-1}P_r}{OP_r} = \frac{P_{r-1}D}{OA} \text{ or } P_{r-1}D = \frac{P_{r-1}P_r}{OP_r}.$$

Thus,

$$EF = \frac{P_{r-1}P_r}{OP_{r-1}OP_r} \simeq \frac{P_{r-1}P_r}{OP_r^2} = \frac{\delta}{1 + AP_r^2} = \frac{\delta}{1 + r^2\delta^2}.$$

Since arc $EG \simeq EF \simeq \dfrac{\delta}{1 + r^2\delta_r^2}$, $\dfrac{1}{8}$ arc of circle is

$$\frac{\pi}{4} = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{\delta}{1 + r^2\delta^2}. \tag{14}$$

Of course, a clear idea of limits did not exist at that time so that the relation was understood in an intuitive sense only. To evaluate the limit, Jyesthadeva uses two lemmas. One is the geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots .$$

Jyesthadeva says that the expansion is obtained on iterating the following procedure:

$$\frac{1}{1+x}=1-x\left(\frac{1}{1+x}\right)=1-x\left(1-x\left(\frac{1}{1+x}\right)\right).$$

The other result is that

$$S_n^{(p)} \equiv 1^p+2^p+\cdots+n^p \sim \frac{n^{p+1}}{p+1} \quad \text{for large } n. \tag{15}$$

A sketch of a proof is given by Jyesthadeva. He notes first that

$$nS_n^{(p-1)}=S_n^{(p)}+S_1^{(p-1)}+S_2^{(p-1)}+\cdots+S_{n-1}^{(p-1)}. \tag{16}$$

This is easy to verify. Relation (16) is also contained in the work of the tenth century Arab mathematician Alhazen, who gives a geometrical proof in the Greek tradition[18]. He uses it to evaluate $S_n^{(3)}$ and $S_n^{(4)}$ which occur in a problem about the volume of a certain solid of revolution. *Yuktibhasa* shows that for $p=2,3$

$$S_1^{(p-1)}+S_2^{(p-1)}+\cdots+S_{n-1}^{(p-1)} \sim \frac{S_n^{(p)}}{p}, \tag{17}$$

and then suggests that by induction the result will be true for all values of p. Once this is granted, it follows that if

$$S_n^{(p-1)} \sim \frac{n^p}{p},$$

then by (16) and (17),

$$nS_n^{(p-1)} \sim S_n^{(p)}+\frac{S_n^{(p)}}{p} \quad \text{or} \quad S_n^{(p)} \sim \frac{n^{p+1}}{p+1},$$

and (15) is inductively proved.

We now note that (14) can be rewritten, after expanding $1/(1+r^2\delta^2)$ into a geometric series, as

$$\begin{aligned}\frac{\pi}{4} &= \lim_{n\to\infty}\left[\delta\sum_{r=1}^{n}1-\delta^3\sum_{r=1}^{n}r^2+\delta^5\sum_{r=1}^{n}r^4-\cdots\right]\\ &= \lim_{n\to\infty}\left[1-\frac{1}{n^3}\sum_{r=1}^{n}r^2+\frac{1}{n^5}\sum_{r=1}^{n}r^4-\cdots\right]\\ &= 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots,\end{aligned}$$

where we have used relation (15) and the fact that $\delta=1/n$. Now consider the approximation (12) and its application to the transformation of series. Suppose that

$$\sigma_n=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\pm\frac{1}{n}\mp f(n+1),$$

where $f(n+1)$ is a rational function of n which will make σ_n a better approximation of $\pi/4$ than the nth partial sum S_n. Changing n to $n-2$ we get

[18]See *The Historical Development of the Calculus* (mentioned in footnote 1), p. 84. Alhazen is the latinized form of the name Ibn Al-Haytham (c. 965–1039).

$$\sigma_{n-2} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots \mp \frac{1}{n-2} \pm f(n-1).$$

Subtracting the second relation from the first,

$$\pm u_n = \sigma_n - \sigma_{n-2} = \pm \frac{1}{n} \mp f(n+1) \mp f(n-1). \tag{18}$$

Then

$$\begin{aligned}\sigma_n &= \sigma_{n-2} \pm u_n \\ &= \sigma_{n-4} \mp u_{n-2} \pm u_n \\ &= \cdots = \sigma_1 - u_3 + u_5 - u_7 + \cdots \pm u_n \\ &= 1 - f(2) - u_3 + u_5 - u_7 + \cdots u_n.\end{aligned}$$

It is clear that

$$\lim_{n\to\infty} \sigma_n = \frac{\pi}{4}$$

and therefore

$$\frac{\pi}{4} = 1 - f(2) - u_3 + u_5 - u_7 + \cdots. \tag{19}$$

Thus, we have a new series for $\pi/4$ which depends on how the function $f(n)$ is chosen. Naturally, the aim is to choose $f(n)$ in such a way that (19) is more rapidly convergent than (1). This is the idea behind the series (13). Now equation (18) implies that

$$f(n+1) + f(n-1) = \frac{1}{n} - u_n. \tag{20}$$

For (19) to be more rapidly convergent than (1), u_n should be $o(1/n)$, that is, negligible compared to $1/n$. It is reasonable to assume $f(n+1) \simeq f(n-1) \simeq f(n)$. These observations together with (20) imply that $f(n) = 1/2n$ is a possible rational approximation in equation (12). With this $f(n)$, the value of u_n is given by (20) to be

$$u_n = \frac{1}{n} - \frac{1}{2(n+1)} - \frac{1}{2(n-1)} = -\frac{1}{n^3 - n}.$$

Substituting this in (19) gives us (13), which is

$$\frac{\pi}{4} = 1 - \frac{1}{4} + \frac{1}{3^3 - 3} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \cdots.$$

The other series

$$\frac{\pi}{4} = \frac{4}{1^5 + 4 \cdot 1} - \frac{4}{3^5 + 4 \cdot 3} + \frac{4}{5^5 + 4 \cdot 5} - \cdots$$

is obtained by taking $f(n) = \dfrac{n/2}{n^2 + 1}$ in (19).

It should be mentioned that Newton was aware of the correction $f_1(n) = 1/2n$. For in the letter to Oldenburg, referred to earlier, he says, "By the series of Leibniz also if half the term in the last place be added and some other like devices be employed, the computation can be carried to many figures." However, he says nothing about transforming the series by means of this correction.

It appears that Nilakantha was aware of the impossibility of finding a finite series of rational numbers to represent π. In the *Aryabhatiya-bhasya* he writes, "If the diameter, measured using some unit of measure, were commensurable with that unit, then the circumference would not likewise allow itself to be measured by means of the same unit; so likewise in the case where the circumference is measurable by some unit, then the diameter cannot be measured using the same unit."[19]

The *Yuktibhasa* contains a proof of the arctan series also and it is obtained in exactly the same way except that one rectifies only a part of the $1/8$ circle.

It can be shown that if $\pi/4 = S_n + f(n)$, where S_n is the nth partial sum, then $f(n)$ has the continued fraction representation

$$f(n) = \frac{1}{2}\left[\frac{1}{n+}\,\frac{1^2}{n+}\,\frac{2^2}{n+}\,\frac{3^2}{n+}\cdots\right]. \tag{21}$$

Moreover, the first three convergents are

$$f_1(n) = \frac{1}{2n}, \qquad f_2(n) = \frac{n/2}{n^2+1} \quad \text{and} \quad f_3(n) = \frac{(n/2)^2+1}{(n^2+5)n/2},$$

which are the values quoted in (13). Clearly, Nilakantha was using some procedure which gave the successive convergents of the continued fraction (21) but the text contains no suggestion that (20) was actually known to him. This continued fraction implies that

$$\frac{2}{4-\pi} = 2 + \frac{1^2}{2+}\,\frac{2^2}{2+}\,\frac{3^2}{2+}\cdots,$$

which may be compared with the continued fraction of the seventeenth century English mathematician, William Brouncker (1620–1684), who gave the result

$$\frac{4}{\pi} = 1 + \frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots.$$

The third approximation

$$f_3(n) = \frac{(n/2)^2+1}{(n^2+5)n/2}$$

is very effective in obtaining good numerical values for π without much calculation. For example

$$1 - \frac{1}{3} + \cdots - \frac{1}{19} + f_3(20)$$

gives the value of π correct up to eight decimal places.[20] Nilakantha himself gives 104348/33215 which is correct up to nine places. It is interesting that the Arab mathematician Jamshid-al-Kasi, who also lived in the fifteenth century, had obtained the same approximation by a different method.

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[19] See *Geschichte der Mathematik*, p. 169.

[20] These observations concerning the continued fraction expansion of $f(n)$ and its relation to the Indian work and that of Brouncker, and concerning the decimal places in f (20), are due to D. T. Whiteside. See "On Medieval Kerala Mathematics" of footnote 13.

5. Independence of these discoveries.

The question naturally arises of the possibility of mutual influence between or among the discoverers of power series, in particular the series for the trigonometric functions. Because of the lively trade relations between the Arabs and the west coast of India over the centuries, it is generally accepted that mathematical ideas were also exchanged. However, there is no evidence in any existing mathematical works of the Arabs that they were aware of the concept of a power series. Therefore, we may grant the Indians priority in the discovery of the series for sine, cosine and arctangent. Moreover, historians of mathematics are in agreement that the European mathematicians were unaware of the Indian discovery of infinite series.[21] Thus, we may conclude that Newton, Gregory and Leibniz made their discoveries independently of the Indian work. In fact, it appears that yet another independent discovery of an infinite series giving the value of π was made by the Japanese mathematician Takebe Kenko (1664–1739) in 1722. His series is

$$\pi^2 = 4\left[1 + \sum_{n=1}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+1)!}\right].$$ [22]

This series was not obtained from the arctan series and its discussion is therefore not included. However, the independent discovery of the infinite series by different persons living in different environments and cultures gives us insight into the character of mathematics as a universal discipline.

Acknowledgement. I owe thanks to Phil Straffin for encouraging me to write this paper and to the referees for their suggestions.

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[21] See "Patterns of Mathematical Thought in the later Seventeenth Century" of footnote 17. See also A. Weil, "History of Mathematics: Why and How" in *Collected Papers*, Vol. 3 (New York: Springer-Verlag, 1979), p. 435.

[22] See D. E. Smith and Y. Mikami, *A History of Japanese Mathematics* (Chicago: Open Court, 1914). This series was also obtained by the French missionary Pierre Jartoux (1670–1720) in 1720. He worked in China and was in correspondence with Leibniz, but the present opinion is that Takebe's discovery was independent. Leonhard Euler (1707–1783) rediscovered the same series in 1737. A simple evaluation of it can be given using Clausen's formula for the square of a hypergeometric series.

NOTES

Why December 21 Is the Longest Day of the Year

How fundamental ideas from max-min theory explain the location of the extrema of the times of sunrise and sunset.

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Sometime during the twentieth century, astronomy divorced mathematics to pursue a more intimate relationship with physics. Although many mathematicians may wish to learn more about astronomy, they will find that more than a superficial knowledge of physics is required to appreciate the latest discoveries. At least, that is how it has seemed to me. Thus it was with some pleasure that I found that elementary mathematics could be applied to explain an astronomical phenomenon that had intrigued me for some time. I am referring to the fact that the extrema of sunrise and sunset do not occur on the shortest or longest days of the year, but are removed from the solstices by almost two weeks (at latitude 44° North—all the data in this article are for the year 1988 at this latitude). In particular, from the point of view of an afternoon person in the northern hemisphere, the days start getting longer around December 9; that is, on that day the sun starts setting later each day. FIGURE 1 shows sunrise and sunset data over the year; it is evident that the extrema are not at the solstices.

This phenomenon is well known to anyone who keeps track of the time the sun rises and sets. Indeed, someone even raised the question in the *New York Times* [4]; the answer given there is accurate, but not really helpful to someone who does not already understand what is going on. Some analysis of graphs using the most elementary notions of calculus can make it clearer.

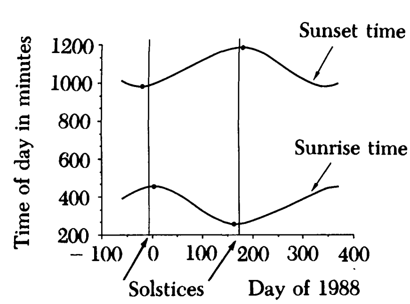

FIGURE 1
The variation in sunrise and sunset times over the year (day 1 is Jan. 1, 1988). From the point of view of an afternoon person, the days start getting longer around December 9.

The key point is to understand at what time of day, by our clocks, the sun is due south. More technically: When is the time of meridian passage of the sun? For reasons to be explained below, this time is not always noon, but varies from noon by as much as 15 minutes. FIGURE 2 shows the complicated behavior of this time, which might be called the time of *solar noon*. Clearly, solar noon is critical to the timing of sunrises and sunsets, since sunset time may be obtained by adding half of the daylight time to the time of solar noon (and, similarly, sunrise time by subtracting). Before turning to that arithmetic, let's try to understand the behavior of the graph in FIGURE 2.

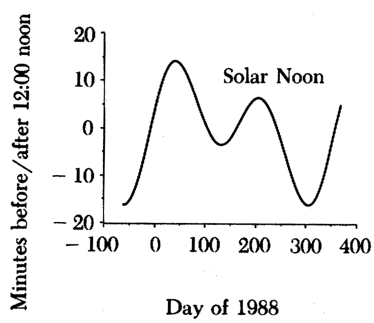

FIGURE 2
The variation in solar noon throughout the year. Note that the greatest advance occurs on December 22, making that day the longest, in terms of time from solar noon to solar noon the next day.

One surprising feature of that graph is that the longest day of the year—measured in the amount of time from solar noon to the next solar noon—is the day from December 22–23: that is when the graph has the steepest slope. Can it be only a coincidence that this longest solar day occurs so near the day commonly considered to be the shortest day of the year?

To understand the solar noon graph it is best to consider its derivative, the solar day-length graph (FIGURE 3). Recall the oft-posed puzzle that asks how many revolutions a penny will make when rolled completely around a stationary penny. The answer, surprising to many, is two. For the same reasons, the earth does not rotate 360° in a day, but must rotate an extra 360/365 of a degree (see FIGURE 4). One way

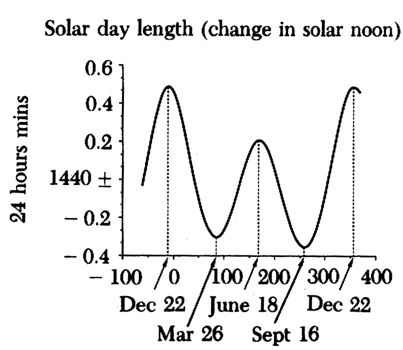

FIGURE 3
The amount of time between consecutive solar noons varies by as much as 30 seconds over the year. However, the effect on the time of solar noon is a cumulative one, so this small variation causes a larger variation in the time of solar noon.

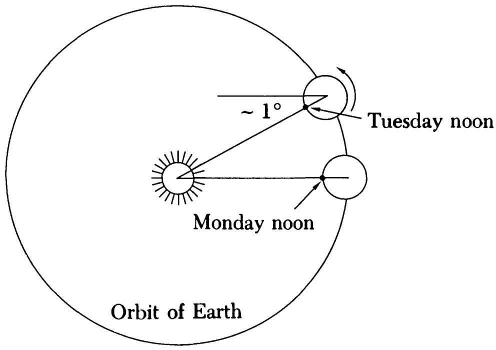

FIGURE 4
In one day the earth spins not 360°, but approximately 361°, so as to catch up to the shortfall caused by its progress around the sun.

of looking at this additional rotation is to interpret it as the amount of spinning the earth does while making up the shortfall caused by the fact that, from the point of view of an observer on earth, the sun has moved. Note that FIGURE 4 is not completely accurate because the earth orbits the sun in an ellipse, and the amount of this orbit travelled by the earth in one day is not constant over the year. In particular, Kepler's Second Law—the earth sweeps out equal areas in equal times—implies that the extra rotation of the earth is greatest when the earth is closest to the sun (*perihelion*), which happens on January 4. This variation is relevant to our discussion, but only in a small way. The sunrise/sunset phenomenon and the general shape of the graphs of solar noon and solar day length would be the same even if the earth's orbit were a circle centered at the sun!

There is a much more important reason why the explanation of FIGURE 4 is inaccurate, one that requires a three-dimensional view. Consider FIGURE 5, where we have switched to an earth-centered view of the universe—more natural for questions of observational astronomy—with North pointing upward. In this figure the earth's equator and the sun's orbit (the *ecliptic*) have been projected onto a large sphere (called the *celestial sphere*). This projection turns the ellipse of the sun's orbit into a circle; note that the ecliptic is tilted at an angle of about $23\frac{1}{2}°$, which is what causes seasons on earth. Now, consider what happens between one solar noon and the next. During that day the sun moves along the ecliptic a certain amount $\Delta\lambda$ (which is, to a first approximation, a constant equal to $\frac{360°}{365}$). To make up for the shortfall engendered by this motion, the earth must spin more than 360° to reach the next noon. But additional earth-spin is measured along the equator, that is, a degree of earth-spin causes a 1° increase in α. Thus the question facing us is How much $\Delta\alpha$ is necessary to yield the required value of $\Delta\lambda$?

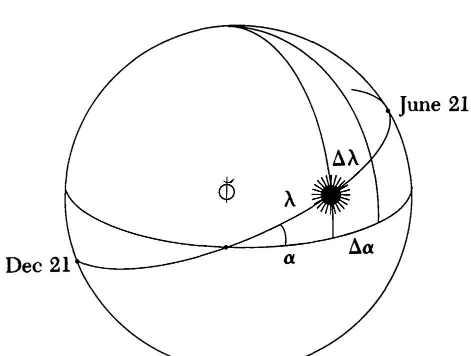

FIGURE 5
The amount of additional rotation needed by the earth to make a full solar day is indicated by $\Delta\alpha$. How much is needed to make up $\Delta\lambda$, the daily shortfall caused by the sun's motion, varies during the year.

The important point to note is that a degree of $\Delta\alpha$ may correspond to more than a degree of $\Delta\lambda$. Near March 21 the geometry is roughly that of a right-angled triangle with $\Delta\lambda$ being the hypotenuse and $\Delta\alpha$ one of the sides; but around December 21 and June 21 the motion of the sun is parallel to the equator, so that $\Delta\lambda$ equals $\Delta\alpha$. As a consequence—keeping in mind that $\Delta\lambda$ is almost constant through the year—more $\Delta\alpha$, that is, more earth-rotation, is needed to make up a solar day during the high and low points of the sun's yearly journey around the earth. This explains why the longest solar days of the year occur at the time of the solstices.

The exact relationship between α (called the *right ascension* of the sun) and λ (the *ecliptic longitude* of the sun) is mathematically very simple: $\tan\alpha = \tan\lambda \cos 23\frac{1}{2}°$. This can be proved by looking at a certain right-angled triangle with a $23\frac{1}{2}°$ angle whose adjacent side has length $\tan\alpha$ and whose hypotenuse has length $\tan\lambda$ (details left as an exercise). It follows that at the equinoxes a degree of $\Delta\alpha$ corresponds to 1.09° of $\Delta\lambda$; as mentioned, at the solstices a degree of $\Delta\alpha$ corresponds to a degree of $\Delta\lambda$.

The preceding does not completely explain the shape of the solar-day-length graph. The varying distance of the earth from the sun during the year means that $\Delta\lambda$ is not constant, as we have assumed. This variation skews the day-length graph toward January 4, the day of perihelion; that is, $\Delta\lambda$, and hence $\Delta\alpha$, are greater around that day, which explains why the two local maxima in FIGURE 3 are not of the same height.

Although the longest solar day differs from 24 hours by only about a half a minute, the effect on the time of solar noon is cumulative, explaining the larger discrepancies between solar noon and 12:00 by our clocks.

Finally, we can turn to the sunrise/sunset question. As we have pointed out, for sunrise it suffices to subtract the half-daylight time from the time of solar noon. The half-daylight graph is given in FIGURE 6; no surprises here—December 21 has the least amount of daylight and June 21 the most. If we subtract this time from solar noon, the resulting graph will have its extrema at the points of horizontal slope. But these points correspond to days when the slope of the solar noon graph exactly equals the slope of the half-daylight graph (and thus the two slopes cancel upon subtraction). This explains what is special about January 4 for sunrises and December 9 for sunsets: those are the days (at latitude 44°) where certain slopes are equal or are negatives of each other. FIGURE 7 shows the equal-slope day that determines the latest sunrise. The situation for sunset is similar.

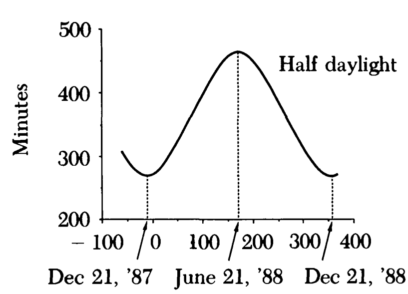

FIGURE 6
The amount of (half) daylight over the year is as we expect, least at the winter solstice, and greatest at the summer solstice. Comparing the slope of this graph with the slope of the solar-day-length graph (FIGURE 3) explains the asymmetry of the earliest and latest sunrises and sunsets.

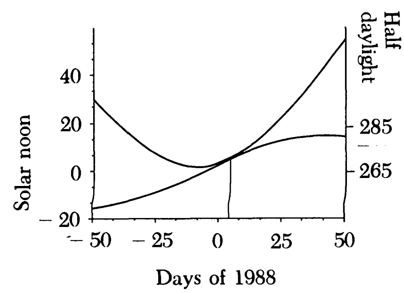

FIGURE 7
Lowering the half-daylight graph shows clearly that the point of tangency—and, therefore, the day of latest sunrise—is several days later than the winter solstice.

The graph of solar noon (Figure 2) is also helpful in understanding the analemma, which is the curve obtained by plotting the sun's position in the sky at noon each day. Of course, the height of the sun at noon goes up and down with the seasons, but because of the difference between our noon and solar noon, there is left-right motion as well. Combining some elementary trigonometry with the data from Figure 2 and some additional data on the sun's height each day yields the image of Figure 8, where 45 daily positions of the sun at noon are diagrammed. See [2] for more on the use of this curve and how it changes over the centuries.

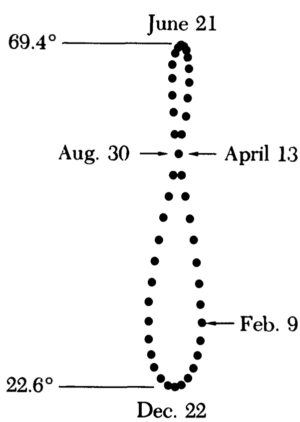

FIGURE 8
The position of the sun in the sky at earth-noon (and at 44° latitude). This curve is called the analemma. Because solar noon is latest on February 9, that is the day when the sun's position is farthest to the right.

The graphs in this article were prepared from data generated by *The Floppy Almanac*, a program for IBM-PCs that computes the astronomical data available in printed form in [3]. The data were transferred to a Macintosh where the Cricket Graph and Cricket Draw programs were used to produce the diagrams that appear in the paper. A good reference for the often complicated mathematics of observational astronomy is the book by Robin Green [1]. I am grateful to astronomer Suzan Edwards and sunset-watcher Peter Gagarin for their interest and help.

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2. Bernard M. Oliver, The shape of the analemma, *Sky and Telescope*, July, 1972, 20–22.
3. *The Astronomical Almanac for the Year 1987*, U.S. Naval Observatory, 1986.
4. Q&A, *The New York Times*, March 15, 1988, p. C6.

A Letter of G. H. Hardy

Among the papers of George Pólya were a letter and postcard Pólya received from G. H. Hardy containing a rather unusual evaluation of poets. Apparently Hardy often amused himself by making up lists or rankings; the Pólya papers contain other examples of this. The present letter, undated but probably written in the late 1920's *or early* 1930's, *challenges Pólya's wife, Stella, to identify an American poet, Ella Wheeler Wilcox. Since Stella Pólya was Swiss, this was a curious fact to expect her to know. When asked in the* 1980's *about the letter, Mrs. Pólya said she thought the challenge was perhaps really intended for Mrs. G. D. Birkhoff. The Pólya papers are in the Archives of Stanford University.*—Editor

The letter reads:

Trinity College
Cambridge
30 Nov.

Dear Pólya

This is merely to acknowledge the MS.

I am giving it to White. It is possible that he may prefer to put it in the *Proc. Camb. Phil. Soc.*, which is just about to open a new series with a better page and style: I told him I was sure that you would not mind whether it was this or the Journal.

I have made quite a lot of red ink marks on it, and must get one of my pupils to make a fair copy.

The theory about my sister is picturesque and, though quite untrue, may well become the official version. Surely it is a little remarkable that (unless your friends are English) they should be able to identify the quotation [it is extremely familiar in English, but not in its context]. I should not have expected Browning to be known at all outside England. He was once "all the rage," but his day, even here, is over and he has no "universal" appeal like Byron or Dickens. When I was a schoolboy, it was necessary to admire him.

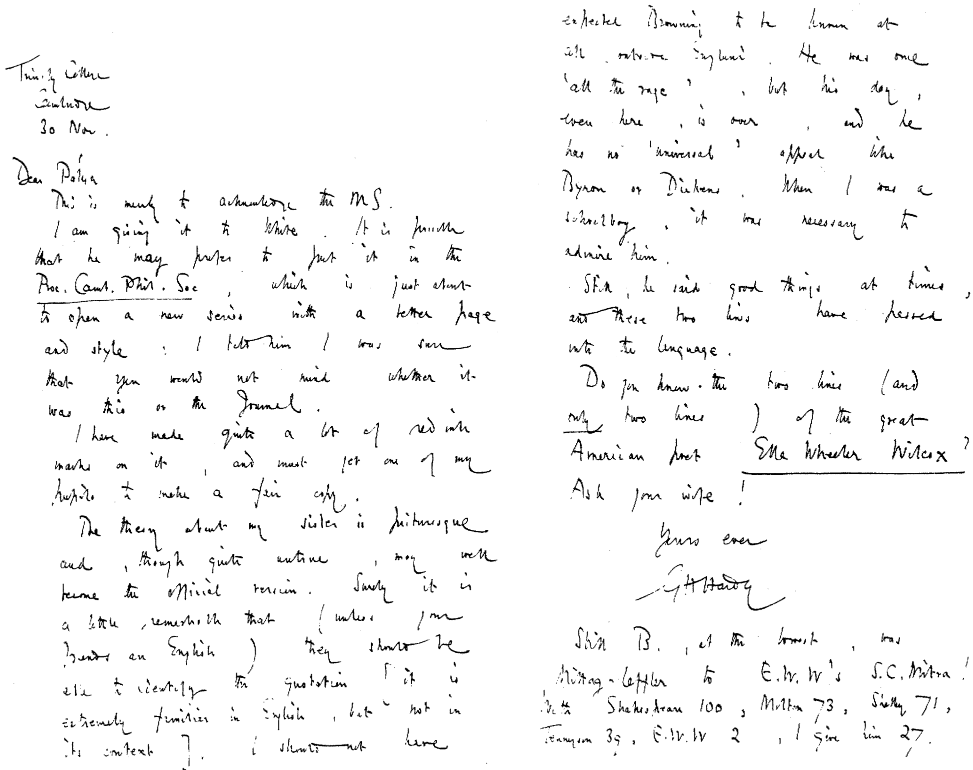
Still, he said good things at times, and these two lines have passed into the language.

Do you know the two lines (and *only* two lines) of the great American poet *Ella Wheeler Wilcox*? Ask your wife!

Yours ever,

G. H. Hardy

Still B., at the lowest, was Mittag-Leffler to E. W. W.'s S. C. Mitra! With Shakespeare 100, Milton 73, Shelley 71, Tennyson 39, E. W. W. 2, I give him 27.



Trinity College
Cambridge
30 Nov.

Dear Polya
This is meant to acknowledge the MS.
I am giving it to White. It is possible that he may prefer to put it in the Proc. Camb. Phil. Soc, which is just about to open a new series with a better page and style: I told him I was sure that you would not mind whether it was this or the Journal.
I have made quite a lot of red ink marks on it, and must get one of my pupils to make a fair copy.
The theory about my sister is picturesque and, though quite untrue, may well become the official version. Surely it is a little remarkable that (unless your friends are English) they should be able to identify the quotation [it is extremely familiar in English, but not in its context]. I should not have expected Browning to be known at all outside England. He was once 'all the rage', but his day, even here, is over, and he has no 'universal' appeal like Byron or Dickens. When I was a schoolboy, it was necessary to admire him.
Still, he said good things at times, and these two lines have passed into the language.
Do you know the two lines (and only two lines) of the great American poet Ella Wheeler Wilcox? Ask your wife!

Yours ever
G H Hardy

Still B., at the lowest, was Mittag-Leffler to E.W.W.'s S.C. Mitra! With Shakespeare 100, Milton 73, Shelley 71, Tennyson 39, E.W.W 2, I give him 27.

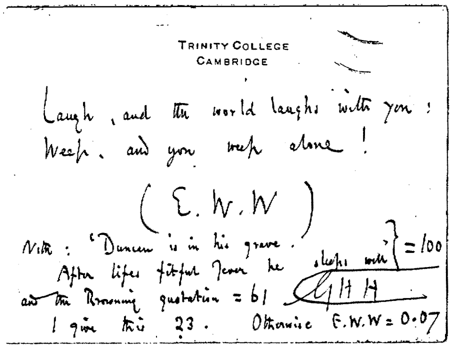
The postcard that followed identified Ella Wheeler Wilcox's most famous line:

Trinity College
Cambridge

Laugh, and the world laughs with you.
Weep, and you weep alone.
(E. W. W.)
With: "Duncan is in his grave,
After life's fitful fever he sleeps well" = 100

and the Browning quotation = 61
I give this 23. Otherwise E. W. W. = 0.07.

G. H. H.



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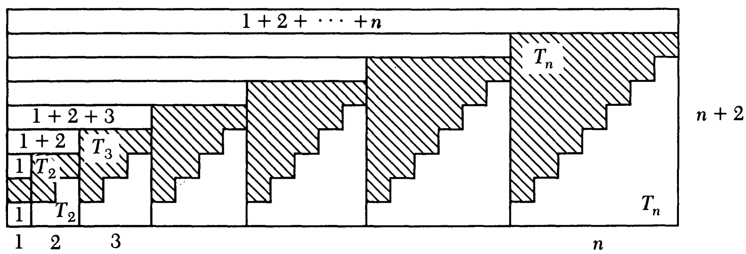
Proof without Words: Sums of Triangular Numbers

$$T_n = 1 + 2 + \cdots + n \Rightarrow T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$$



$$3(T_1 + T_2 + \cdots + T_n) = (n+2) \cdot T_n$$

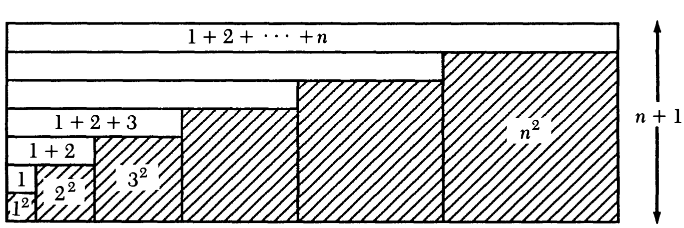
$$T_1 + T_2 + \cdots + T_n = \frac{(n+2)}{3} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

—MONTE J. ZERGER
Adams State College
Alamosa, CO 81102

Proof without Words Corollary: Sums of Squares

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$



$$1^2 + 2^2 + \cdots + n^2 + (T_1 + T_2 + \cdots + T_n) = T_n \cdot (n+1)$$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)^2}{2} - \frac{n(n+1)(n+2)}{6} = \frac{n(n+1)(2n+1)}{6}$$

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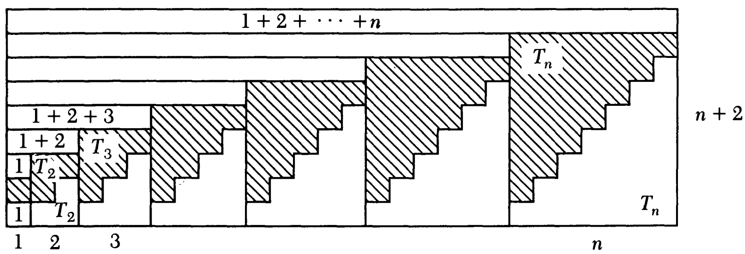

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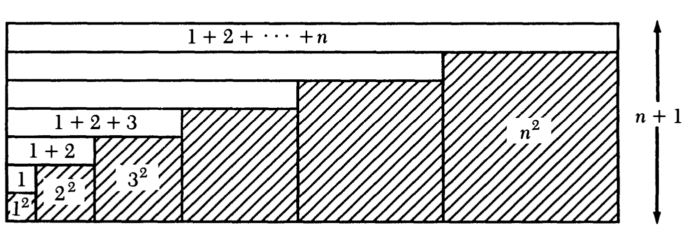

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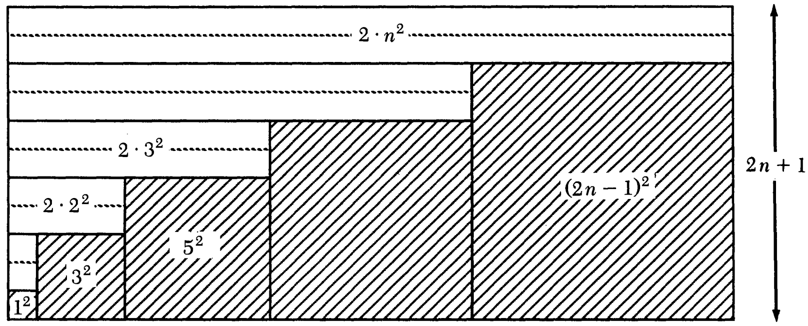

$$1^2+3^2+\cdots+(2n-1)^2+2(1^2+2^2+\cdots+n^2)=n^2\cdot(2n+1)$$

$$1^2+3^2+\cdots+(2n-1)^2=n^2(2n+1)-\frac{n(n+1)(2n+1)}{3}=\frac{n(2n-1)(2n+1)}{3}$$

—Roger B. Nelsen
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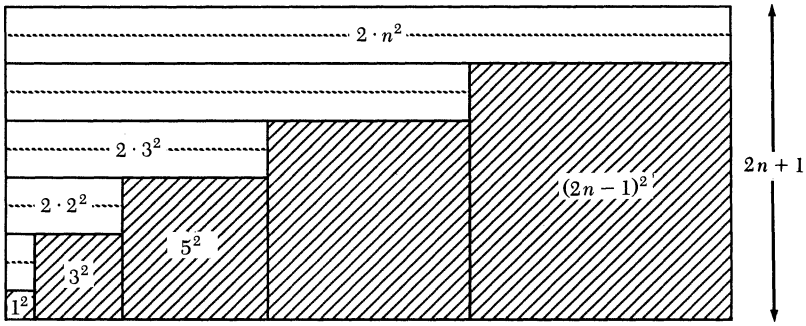
Cyclic Groups and the Generation of La Loubère Magic Squares

DONALD H. ECKHARDT
Air Force Geophysics Laboratory
Bedford, MA 01731

The magical squares, however wonderful soever they may seem, are what I cannot value myself upon, but am rather ashamed to have it known I have spent any part of my time in employment that cannot possibly be of any use to myself or others.

Benjamin Franklin [1]

In 1687 Isaac Newton published his first edition of *Philosophiae Naturalis Principia Mathematica*. The *Principia* is as much a work of natural philosophy as of mathematics, so mathematicians have to share the tercentenary jubilation with physicists, astronomers, and other natural scientists. For mathematicians I offer an anniversary you can celebrate by yourselves: In 1687, King Louis XIV dispatched Simon de La Loubère as his extraordinary ambassador to the King of Siam. On his return, La Loubère, one of the great dilettantes of his time, wrote a scholarly account of his voyage via India and his short stay in Siam. He communicated, among other things, the official rules that the Siamese used to track the sun and the moon, and a technique that the Indians used to construct magic squares. The astronomical rules were initially of such import that Jean Dominique Cassini, the director of the Paris Observatory, cited them in one of his own publications. Nevertheless, they were only of temporary value for the revolution that Newton ignited eventually made Siamese



$$1^2 + 3^2 + \cdots + (2n-1)^2 + 2(1^2 + 2^2 + \cdots + n^2) = n^2 \cdot (2n+1)$$

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A *magic square* is an $n \times n$ square array of all the integers 1 through n^2 that are arranged so that each row, column and diagonal sums to $n(n^2+1)/2$. The La Loubère technique may be used to construct magic squares only for odd *order* (number of columns or rows)—1×1 (trivial), 3×3, 5×5, etc.—square arrays. Throughout this paper, n will be assumed to be an odd number greater than 1. To use the La Loubère technique for an $n \times n$ array, first consider the array to be inscribed on a torus, so that the first row is immediately below the nth row, and the first column is immediately to the right of the nth column; the cells in all rows and columns recycle modulo n. Start in the center cell of the first row, and insert a 1. Then proceed along a diagonal (chess bishop's) path "northeast" (up and to the right), inserting the successive integers (2, 3, 4, etc.) into the cells traversed. Whenever the path is obstructed by an already occupied cell, go "south" (down) one cell, and continue the "northeast" path from there. For $n = 5$, the resulting magic square looks like this:

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

In this example, all rows, columns, and the two (principal) diagonals sum to $5(5^2+1)/2 = 65$.

In 1957, I was working as an exploration seismologist in South America. My father wrote me a letter noting that if you start with the *primary* square, e.g. (for $n = 5$),

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

and consider the operator $\mathscr{R}_n$ that maps the primary square into the La Loubère magic square, then $\mathscr{R}_n$ is the generator of a cyclic group. For the 5×5 square, $\mathscr{R}_5$ maps with the permutation

$$(13)(2\ 24)(10\ 16)(1\ 17\ 12\ 6\ 23\ 25\ 9\ 14\ 20\ 3)$$
$$(4\ 8\ 7\ 5\ 15\ 22\ 18\ 19\ 21\ 11).$$

The order of $\mathscr{R}_5$ is the least common multiple, 10, of the orders of these cycles; that is, $q = 10$ is the least positive integer for which $\mathscr{R}_5^q = \mathscr{I}$, the identity operator. The order of $\mathscr{R}_5$ is 10; the order of $\mathscr{R}_3$ is 4; and the order of $\mathscr{R}_7$ is 8. My father wanted to know the relationship between the order n of the magic square and the order q of the cyclic group generated by $\mathscr{R}_n$. (Because n is finite, the group is finite and q is finite.) For large n, could q be determined without actually having to construct the $n \times n$ magic square? (Answer below: *sometimes*.) In a table relating n and q, would there be any clear pattern? (Answer below: *not really*.)

I considered my father's mathematical questions annoying because I dared not fail his high expectations. I felt obliged to answer all of his questions, and it was not always easy. After my father died in 1985, I discovered that he had saved many of my letters and they included my solutions to various mathematical problems that he had posed. One long letter contained my examination of La Loubère magic square mappings, which I had worked out somewhere in Venezuela. I present it now.

Suppose the $n \times n$ primary square is modified by writing it in the base n, starting with 0 instead of 1. For its corresponding magic square, each row, column and diagonal should sum to $n(n^2-1)/2$. The 5×5 primary square is then

00	01	02	03	04
10	11	12	13	14
20	21	22	23	24
30	31	32	33	34
40	41	42	43	44

The two diagonals sum to $220_5 = 60_{10} = 5(5^2-1)/2$. If the entries in the first row are shifted to the right by two cells, those in the second row to the right by one cell, those in the third row by none, those in the fourth row to the left by one cell, and those in the fifth row to the left by two cells, the square is transformed (on the torus) to

03	04	00	01	02
14	10	11	12	13
20	21	22	23	24
31	32	33	34	30
42	43	44	40	41

The operator that performs this mapping is designated $\mathscr{H}_5$. Now there are both first and second digits 0 through 4 in each column, none repeated. The columns sum to 220_5, as do the diagonals which have been symmetrically altered so that their sums are unchanged. The sums of the rows remain unchanged, of course, but the entries in the columns can next be shifted so that the rows add up to 220_5 as well, without changing the sums of the columns and diagonals. The operator $\mathscr{V}_5$, that effects this final shift to end up with a magic square, works as follows: the entries in the first column are shifted down by two cells, those in the second column down by one cell, those in the third column by none, those in the fourth column up by one cell, and those in the fifth column up by two cells. The square is then transformed to

31	43	00	12	24
42	04	11	23	30
03	10	22	34	41
14	21	33	40	02
20	32	44	01	13

Now there are both first and second digits 0 through 4 in each column and in each row, and the diagonal sums remain unchanged. Converting this square back to decimal base, and adding 1 to each number, it is the 5×5 La Loubère magic square. That is, $\mathscr{V}_5\mathscr{H}_5 = \mathscr{R}_5$. For the general case, the operator $\mathscr{H}_n$ shifts the entries in row i to the right by $(n+1)/2 - i$ cells, and the operator $\mathscr{V}_n$ shifts the entries in column j down by $(n+1)/2 - j$ cells. Then $\mathscr{V}_n\mathscr{H}_n = \mathscr{R}_n$ is an operator that reorders the base n integers 00 through $(n-1)(n-1)$ of the modified primary square so that there are both first and second digits 0 through $(n-1)$ in each column and in each row, and the diagonal sums remain unchanged. Converting the resulting square back to decimal base, and adding 1 to each number, it is the $n \times n$ La Loubère magic square. This approach applies for constructing all La Loubère magic squares and, with additional shifting operators, the approach can easily be extended for constructing magic cubes, magic tesseracts, and so forth. (Before you try that, I suggest that you re-read Benjamin Franklin's comment quoted at the beginning of this note.) There are other operators of the form $\mathscr{V}_n^i\mathscr{H}_n^j$ or $\mathscr{H}_n^i\mathscr{V}_n^j$, i and j being integers, that also generate magic squares from primary squares. With the operator $\mathscr{V}_n^2\mathscr{H}_n = \mathscr{V}_n\mathscr{R}_n$, for example, the bishop's path of the La Loubère technique is replaced by a sequence of knight's moves (two cells up and one cell to the right).

Because the central cell of each square remains unchanged by $\mathscr{R}_n$, an appropriate way for assigning coordinates to each cell of an odd order square is to put the central cell at the origin of a cartesian system with unit spacing. For the 5×5 square, the (x, y) coordinates are (mod 5):

3, 2	4, 2	0, 2	1, 2	2, 2
3, 1	4, 1	0, 1	1, 1	2, 1
3, 0	4, 0	0, 0	1, 0	2, 0
3, 4	4, 4	0, 4	1, 4	2, 4
3, 3	4, 3	0, 3	1, 3	2, 3

In general, $\mathscr{H}_n$ maps the entry in cell (x, y) into cell $(x+y, y) \pmod n$ and $\mathscr{V}_n$ maps the entry in cell (x, y) into cell $(x, x+y) \pmod n$. Thus $\mathscr{R}_n = \mathscr{V}_n\mathscr{H}_n$ maps the entry in cell (x, y) into cell $(x+y, x+2y) \pmod n$. Let

$$\mathbf{R} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\mathbf{R}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+2y \end{bmatrix}.$$

Therefore, the cyclic groups generated by $\mathscr{R}_n$ and $\mathbf{R} \pmod n$ are isomorphic, and

LEMMA. *$\mathscr{R}_n^k = \mathscr{I}$ if and only if $\mathbf{R}^k \equiv \mathbf{I} \pmod n$.*

The matrix $\mathbf{R}^k$ has an interesting relationship with the Fibonacci series [5, 6, 7]. The Fibonacci number u_j is determined by the definitions $u_0 = 0$ and $u_1 = 1$, and the recursion

$$u_{j+1} = u_j + u_{j-1}. \tag{1}$$

Let

$$\mathbf{U}_j = \begin{bmatrix} u_{j-1} & u_j \\ u_j & u_{j+1} \end{bmatrix}.$$

Then, using (1) and noting that

$$\mathbf{U}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\mathbf{U}_j = \mathbf{U}_1\mathbf{U}_{j-1} = \mathbf{U}_1^2\mathbf{U}_{j-2} = \cdots = \mathbf{U}_1^j.$$

The determinant of $\mathbf{U}_1$ is $\det(\mathbf{U}_1) = -1$, so $\det(\mathbf{U}_j) = \det^j(\mathbf{U}_1) = (-1)^j$; that is:

$$u_{j-1}u_{j+1} - u_j^2 = (-1)^j. \tag{2}$$

Now the relationship between $\mathbf{R}^k$ and the Fibonacci series becomes apparent: $\mathbf{R} = \mathbf{U}_2$, and so $\mathbf{R}^k = \mathbf{U}_{2k}$, and $\det(\mathbf{R}^k) = \det(\mathbf{R}) = 1$. Thus,

$$\mathbf{R}^k = \begin{bmatrix} u_{2k-1} & u_{2k} \\ u_{2k} & u_{2k+1} \end{bmatrix} \quad \text{and} \quad \mathbf{R}^{-k} = \begin{bmatrix} u_{2k+1} & -u_{2k} \\ -u_{2k} & u_{2k-1} \end{bmatrix}. \tag{3}$$

By the Lemma, $\mathscr{R}_n^k = \mathscr{I}$ if and only if $u_{2k-1} \equiv u_{2k+1} \equiv 1 \pmod n$ and $u_{2k} \equiv 0 \pmod n$. If $u_{2k} \equiv 0 \pmod n$ and $u_{2k-1} \equiv 1 \pmod n$, then substituting $2k$ for j in (1) shows that the third condition, $u_{2k+1} \equiv 1 \pmod n$, is also fulfilled. What has been proven is:

THEOREM 1. *$n | u_{2k}$ and $n | u_{2k-1} - 1$ are necessary and sufficient conditions for $\mathscr{R}_n^k = \mathscr{I}$.*

A corollary to Theorem 1 is the result that every prime divides some Fibonacci number (because $2 | u_3$, n can be any odd prime, and the order of $\mathscr{R}_n$ is finite).

Suppose that $n | u_{2k}$, but that $n \nmid u_{2k-1} - 1$. Then setting $2k = j-1, j+1$ in (2) gives $u_{2k-1}^2 \equiv u_{2k+1}^2 \equiv 1 \pmod n$. What follows is

$$\mathbf{R}^k \equiv \begin{bmatrix} u_{2k-1} & 0 \\ 0 & u_{2k+1} \end{bmatrix} \pmod n, \quad \text{and so } \mathbf{R}^{2k} \equiv \begin{bmatrix} u_{2k-1}^2 & 0 \\ 0 & u_{2k+1}^2 \end{bmatrix} \equiv \mathbf{I} \pmod n.$$

By Theorem 1, $\mathscr{R}_n^{2k} = \mathscr{I}$. Next suppose only that $\mathscr{R}_n^{2k} = \mathscr{I}$. Then, using $\mathbf{R}^k$ and $\mathbf{R}^{-k}$ from (3),

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{2k+1} & -u_{2k} \\ -u_{2k} & u_{2k-1} \end{bmatrix} \equiv \mathbf{R}^{2k}\mathbf{R}^{-k} \equiv \mathbf{R}^k \equiv \begin{bmatrix} u_{2k-1} & u_{2k} \\ u_{2k} & u_{2k+1} \end{bmatrix} \pmod n,$$

which, because $2 \nmid n$, requires that $u_{2k} \equiv 0 \pmod n$. Another theorem has been proven:

THEOREM 2. *$n | u_{2k}$ is a necessary and sufficient condition for $\mathscr{R}_n^{2k} = \mathscr{I}$.*

The theorems provide criteria for searching for the order q of $\mathscr{R}_n$ from a table of prime factors of Fibonacci numbers. Assume that the index k of u_k is nonnegative. If $q > k-1$ and $n | u_{2k}$ then either (Case 1): $n | u_{2k-1} - 1$ and $q = k$; or (Case 2): $n \nmid u_{2k-1} - 1$ and $q = 2k$. (In Case 2, $k < q < 2k$ is not possible because $\mathscr{R}_n^{2k} = \mathscr{I}$ would then require that $\mathscr{R}_n^{2k-q} = \mathscr{I}$; then $2k - q < k < q$, and $j = q$ would not be the least positive integer for which $\mathscr{R}_n^j = \mathscr{I}$.) Below is a table of even index Fibonacci numbers and their factors. Each factor p^m (p an odd prime and m a positive integer) of u_{2k} for which $p^m | u_{2k-1} - 1$ is in **boldtype**. From the table you can see, for example, that 3, 7, and 21 are divisors of u_{16} and of $u_{15} - 1$, whereas 47, 141, 329, and 987 are divisors of u_{16} but not of $u_{15} - 1$. By Theorem 1, $\mathscr{R}_n^8 = \mathscr{I}$ for $n = 3$, 7 and 21; and by Theorem 2, $\mathscr{R}_n^{16} = \mathscr{I}$ for $n = 3, 7, 21, 47, 141, 329$, and 987. Because 7, 21, 47, 141, 329, and 987 are not divisors of any u_{2k} and $u_{2k-1} - 1$, $k < 8$, the order of $\mathscr{R}_7$ and $\mathscr{R}_{21}$ is 8, and the order of $\mathscr{R}_{47}$, $\mathscr{R}_{141}$, $\mathscr{R}_{329}$ and $\mathscr{R}_{987}$ is 16. Because 3 is also a divisor of u_8 and $u_7 - 1$ but not of any u_{2k} and $u_{2k-1} - 1$, $k < 4$, the order of $\mathscr{R}_3$ is 4.

k	u_{2k-1}	u_{2k}
1	1	$1 = 1$
2	2	$3 = 3$
3	5	$8 = 8$
4	13	$21 = \boldsymbol{3} \times 7$
5	34	$55 = 5 \times \boldsymbol{11}$
6	89	$144 = 16 \times 9$
7	233	$377 = 13 \times \boldsymbol{29}$
8	610	$987 = \boldsymbol{3} \times \boldsymbol{7} \times 47$
9	1597	$2584 = 8 \times 17 \times \boldsymbol{19}$
10	4181	$6765 = 3 \times \boldsymbol{5} \times \boldsymbol{11} \times 41$
11	10946	$17711 = 89 \times \boldsymbol{199}$
12	28657	$46368 = 32 \times \boldsymbol{9} \times 7 \times 23$
13	75025	$121393 = 233 \times \boldsymbol{521}$
14	196418	$317811 = 3 \times \boldsymbol{13} \times \boldsymbol{29} \times 281$
15	514229	$832040 = 8 \times 5 \times \boldsymbol{11} \times \boldsymbol{31} \times 61$
16	1346269	$2178309 = \boldsymbol{3} \times \boldsymbol{7} \times \boldsymbol{47} \times 2207$
17	3524578	$5702887 = 1597 \times \boldsymbol{3571}$
18	9227465	$14930352 = 16 \times 27 \times \boldsymbol{17} \times \boldsymbol{19} \times 107$
19	24157817	$39088169 = 37 \times 113 \times \boldsymbol{9349}$
20	63245986	$102334155 = \boldsymbol{3} \times \boldsymbol{5} \times 7 \times \boldsymbol{11} \times \boldsymbol{41} \times 2161$
21	165580141	$267914296 = 8 \times 13 \times \boldsymbol{29} \times \boldsymbol{211} \times 421$
22	433494437	$701408733 = 3 \times 43 \times \boldsymbol{89} \times \boldsymbol{199} \times 307$
23	1134903170	$1836311903 = \boldsymbol{139} \times \boldsymbol{461} \times 28657$
24	2971215073	$4807526976 = 64 \times \boldsymbol{9} \times \boldsymbol{7} \times \boldsymbol{23} \times 47 \times 1103$
25	7778742049	$12586269025 = 25 \times \boldsymbol{11} \times \boldsymbol{101} \times \boldsymbol{151} \times 3001$
26	20365011074	$32951280099 = 3 \times \boldsymbol{233} \times \boldsymbol{521} \times 90481$

There is just enough information in the table to create the following table which shows q, the order of $\mathscr{R}_n$, as a function of n over the interval $3 \leqslant n \leqslant 43$:

n	q	n	q	n	q
3	4	17	18	31	15
5	10	19	9	33	20
7	8	21	8	35	40
9	12	23	24	37	38
11	5	25	50	39	28
13	14	27	36	41	20
15	20	29	7	43	44

A concluding remark: I think that combining *magic squares* and *Fibonacci numbers* with *group theory* and *number theory* is nothing short of fun. Why not include *numerology*? (After all, is there any significance in anniversaries of three centuries or three decades to anybody but a numerologist?) I might also add *topology* (remember the torus?) but that would be stretching a point, and in topology you can stretch all sorts of things: inner tubes, pretzels, even tea cups; but you can never stretch a point.

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Pascal's Wager: A Decision-Theoretic Approach

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Pascal's wager was designed to present an argument intended to encourage living a life dedicated to God. As a mathematician, scientist, and philosopher, Pascal thought that even though it may not be possible to determine God's existence beyond the shadow of any doubt, a reasonable person would choose to live his or her life so as to achieve the maximum likely benefit.

The following quote from article 233 of Pascal's *Pensées* [3] demonstrates his game-theoretic approach to this question.

> Let us then examine this point and say, "God is, or He is not." ... What will you wager? ... Let us weigh the gain and loss of wagering that God is ... there is here an infinity of an infinitely happy life to gain, a chance of gain against a finite number of chances of loss, and what you stake is finite ... every player stakes a certainty to gain an uncertainty, and yet he stakes a finite certainty to gain a finite uncertainty, without transgressing against reason ... the uncertainty of the gain is proportioned to the certainty of the stake according to the proportion of the chances of gain and loss. ... And so our proposition is of infinite force, when there is the finite to stake in a game where there are equal risks of gain and loss, and the infinite to gain. This is demonstrable; and if men are capable of truths, this is one. ...

The payoff table given below summarizes Pascal's wager. Notice that although it is in all likelihood quite impossible to quantify the factors needed by the wager, the analysis of this wager still greatly clarifies the issues involved. This clarification is achieved by the simple expedient of expressing the argument in the language of mathematical formalism. This situation is typical of qualitative models of real life situations. Note that this game exhibits the peculiarity that only one player, the human, makes a strategy decision. Games of this type do represent a simpler or degenerate form of the two-person game and are referred to in Bram's book [1] as decision-theoretic and in Gale's book [2] as games against nature.

The clarity and power of mathematical symbolism and analysis was precisely the tool needed to permit an objective evaluation of uncertain situations. Blaise Pascal (1623–1662) and Pierre de Fermat (1601–1665) through an extended correspondence developed the foundations for the then fledging mathematical science of probability theory. Such mathematical tools are essential for the science of modern decision making.

To implement the payoff table for Pascal's wager it is necessary to specify all the parameters or factors involved.

Positive payoffs:

$$J = \text{joys of Heaven}$$

and

$$G = \text{gratifications of life.}$$

Pascal reasoned that since the life after death was eternal, that J would be infinite. He also reasoned that the pleasures of the earthly life are finite, since that life itself is finite. As a consequence G is finite.

Negative payoffs:

$$-A = \text{anguish of Hell}$$

and

$$-B = \text{burden of a disciplined life.}$$

Pascal did not mention these negative payoffs specifically and they might be construed to be the loss of their positive counterparts above, that is, $-A$ might be viewed as a loss of J, and $-B$ as the loss of G. For purposes of generality we will distinguish $-A$ and $-B$ from J and G, respectively.

Notice that there is no need actually to specify or quantify J, G, A and B exactly. It is only necessary to note that J and A are overwhelmingly large payoffs, likely infinite, whereas B and G are finite payoffs that are exceedingly, indeed vanishingly, small in comparison.

A brief comment is in order concerning the infinity of J and A. Pascal clearly declares the infinity of J. He did not remark about A at all. Still if an infinity (or eternity) of Heaven is possible, then an eternity of its opposite, Hell, is also possible and its payoff $-A$ could also be infinite. Now $-A$ might be zero, i.e., no suffering, or it might be the knowledge of the loss of J or it might involve some additional suffering. As will be shown below the magnitude of A will not affect the outcome of Pascal's argument.

An objection to the infinity of J could be raised on the following basis. If eternity can be represented as an infinity of finite time segments, say days, and the daily level of heavenly happiness diminished at a fast enough rate, then the infinite time integral of happiness could be finite, even small. But the recognition that today you were less happy than you were yesterday would be a source of unhappiness. This situation is parallel to the apparently demonstrated diminution over time of the satisfaction of wealth. Thus to be happy in heaven, that daily happiness must at least remain constant and consequently J would be infinite. Note further that not only did Pascal postulate infinite happiness but even more an infinity of infinite happiness.

Observe that in keeping with standard notation for matrix games all payoffs represent either a win or a loss to one of the players. The negative (positive) designation given to the payoffs above indicates that they represent a loss (win) to the only active player, the human.

The truth about God's existence:

$$T1 = \text{God exists in Pascal's sense}$$

and

$$T2 = \text{God does not exist in Pascal's sense.}$$

Pascal's basic argument is that if God exists then

1. there is an eternity of happiness in heaven available versus a brief period of earthly pleasures; and

2. your life on earth will determine whether you receive the joys of heaven or not.

Thus proposition $T1$ includes two additional conditions, namely, human life after death is eternal and God is not indifferent to human behavior. Now $T2$ is the negation of $T1$ and requires some further analysis.

To accomplish this consider the following three conditions:

$P1$ = God exists,
$P2$ = God is not indifferent to human behavior, and
$P3$ = life after death for human beings is eternal.

The following truth table for $P1$, $P2$, and $P3$ will clarify the differences between $T1$ and $T2$.

Case	$P1$	$P2$	$P3$	Comment
1	F	T	F	impossible
2	F	T	T	impossible
3	F	F	F	$T2$
4	F	F	T	$T2$
5	T	F	F	$T2$
6	T	F	T	$T2$
7	T	T	F	$T2$
8	T	T	T	$T1$

In cases 1 and 2 if God does not exist then He cannot be not indifferent. In cases 3 and 5 and 7 there is no eternal human life after death. In cases 4 and 6 it does not matter how you live your life on earth anyway. Thus $T2$ = not $T1$ consists of 5 out of 6 possible cases. Note further that in cases 5, 6 and 7, "God exists" but the other conditions needed to support Pascal's argument are not satisfied. The philosophical and theological implications of cases 3 through 7 will be left unexplored, since the sole purpose of this paper is to demonstrate how Pascal's argument could be formalized. These cases though do illustrate the variety of ways in which $T1$ can be denied.

The human player's strategy:

$$S1 = \text{Live as if } T1 \text{ were true}$$

and

$$S2 = \text{Live as if } T2 \text{ were true}.$$

Strategy $S1$ assumes that it is possible to know how to lead a God-pleasing life. Strategy $S2$ says that the human player chooses to live a life that is not God-pleasing for otherwise the player would have adopted strategy $S1$. Strategy $S2$ forces the player to follow a lifestyle whose choices are not motivated by God's plan as in strategy $S1$, but by the player's plan.

Pascal is asking in his wager for you to bet that a personal (has a plan for your life) and loving (wants you to spend eternity in Heaven in happiness) God exists. Pascal's argument does not include the possibility of any gain after death that might come to the human player in case $T2$ is true. This is because in cases 3, 4, 5, and 6 any possible gain is independent of which strategy $S1$ or $S2$ is chosen, since $P2$ is false. In case 7 things are slightly different. Like case 8, there could be a divine reward for adopting strategy $S1$ and a divine punishment for adopting $S2$, but we assume that the payoff is finite, possibly zero, since there is no eternal life.

A further comment on the assumption of case 7 might be appropriate. Since $P3$ is false, then life after death is finite. This finite duration is probably zero, but it could be nonzero. In the nonzero case, it might happen that incremental rewards (punishments) would occur over an infinite sequence of times converging to a finite limiting time. Depending on the size of the increments and the rate of convergence, the aggregate payoff could be infinite. If this payoff is finite then this is a type of the original case 7, if it is infinite call this case 7a. Admitting this unique case, 7a, derived from this cascade of exceptional circumstances still will not alter the final choice of the dominant strategy. The reason is simply that case 7a's infinite payoffs will force the same choice of dominant strategy as do case 8's infinite payoffs. Also case 8's infinite payoffs dominate case 7a's, since infinite joy (anguish) for eternity is greater (less) than infinite joy (anguish) for a finite time. Although a situation like case 7a is not expressly discussed in Pascal's wager, we see that this will not alter the validity of his argument.

How do we know that there are no after death payoffs to the human player, even if $T1$ is false? To guarantee that no possible after death payoffs have been neglected in case $T2$ is true, we adopt the notation, $Qi =$ the payoff to the human player for adopting strategy Si, for $i = 1, 2$. We refer to these payoffs as quixotic because in general both their source and their nature could be unknown, hence Qi could be positive, negative or zero. Qi would represent a finite payoff in cases 3 and 5 and 7, because $P3$ is false, could represent an infinite payoff in cases 4 and 6, because $P3$ is true, would satisfy $Q1 = Q2$ for cases 3, 4, 5 and 6, because $P2$ is false and would satisfy $Q1 \geqslant 0 \geqslant Q2$ for case 7, because $P2$ is true and hence in this case $Q1$ represents a reward and $Q2$ represents a punishment. The analysis given below will demonstrate that the addition of these quixotic payoff factors will not affect the outcome of Pascal's wager.

Another possible payoff factor that could be added is a reward (punishment) that is appropriate to the human player's behavior. Thus the proportion of J and $-A$ received by the human player would be adjusted to a scale of degrees of "goodness" and "badness." In the context of this payoff table, the proportion of J and $-A$ received by the human player is adjusted to the frequency of playing strategy $S1$ versus $S2$. This suggests that mixed strategies for the human player can represent a "bad" to "good" scale. One difficulty with this approach is that it requires a model for the scale of "goodness" that is applied by God to the human player's performance. Is it relative or absolute? Does God grade on a curve or not? What is a passing grade? The analysis given below will demonstrate that the addition of a graded payoff factor would not affect the outcome of Pascal's wager.

Probabilities:

$$p = \text{the probability that } T1 \text{ is true}, \quad (0 \leqq p \leqq 1)$$

and

$$1 - p = \text{the probability that } T2 \text{ is true}, \quad (0 \leqq 1 - p \leqq 1).$$

Pascal assumed that these were the only probability factors involved.

The payoff table below will show that this choice was correct, since pure strategies for the human player will prove to be optimal, hence mixed strategies and their associated probability factors need not be considered.

Payoff:

TABLE	$T1\,(p)$	$T2\,(1-p)$	Expected Payoff
$S1$	$J-B$	$Q1-B$	$p(J-B)+(1-p)(Q1-B)=pJ-B+(1-p)Q1$
$S2$	$G-A$	$Q2+G$	$p(G-A)+(1-p)(Q2+G)=G-pA+(1-p)Q2$

The payoff entries, $J-B$ and $Q1-B$, represent the net gain, if strategy $S1$ is adopted and similarly, $G-A$ and $Q2+G$, represent the net gain, if strategy $S2$ is adopted.

Notice that $pJ>B$ and $pA>G$, no matter how small p is because J and A would be overwhelmingly large, in fact infinite, thus $pJ-B>0>G-pA$. This exactly supports Pascal's conclusion that the payoff for strategy $S1$ is preferred so long as $p>0$. He also argued that since no one could be absolutely certain that $p=0$, then a prudent human player would choose strategy $S1$.

Notice also that the addition of the quixotic payoff factor not mentioned by Pascal, $(1-p)Qi$, does not alter the central conclusion of his argument, namely, the expected payoff for strategy $S1=pJ-B+(1-p)Q1>G-pA+(1-p)Q2=$ the expected payoff for strategy $S2$, since $Q1\geqslant Q2$, always. Thus even the possible presence of such a payoff factor as Qi does not alter the correctness of Pascal's argument; although the size of $Q2$ could affect the negativity of $G-pA$, in cases 3, 4, 5, and 6, that is, $G-pA+(1-p)Q2$, could be nonnegative, if it was reasonable to argue that quixotic payoff $Q2$ was positively infinite.

A final comment regarding the additivity of the payoff factors in this table is in order, since some payoffs might be infinite. In the case where only J or A or $Q1$ or $Q2$ are infinite, they can be treated as extremely large finite numbers without invalidating the argument, since J and A differ in sign and $(1-p)Q1>(1-p)Q2$.

Conclusion Pascal's argument is not intended to prove that $T1$ is true, but it does show that unless p is 0 or both J and A are not extremely large, then $S1$ is the best strategy. The peculiarity of the strength of the argument is that it does not rely on finding evidence to support the truth of $T1$, rather it relies on the difficulty of proving beyond the shadow of any doubt that $T1$ is false and hence that $T2$ must be true. Thus even the provable indeterminancy of $T1$ cannot deflect the strength of Pascal's argument.

Afterword My aim in this paper has been to transform Pascal's probability model for his wager into a decision-theoretic model that is faithful to his original intent. Both the model described and the analysis presented were intended to preserve the essential features of his wager. Any failure to accomplish this is solely my responsibility and should not reflect on either the beauty or cogency of Pascal's original presentation.

Acknowledgements. Helpful comments by various referees have improved the expository content of the paper.

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Karl Menger and Taxicab Geometry

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In 1952 Karl Menger established a geometry exhibit at the Museum of Science and Industry of Chicago. Accompanying the exhibit was a booklet, entitled *You Will Like Geometry*, in which the term "taxicab geometry" was first used. The name has remained associated with the geometry and the geometry itself has generated interest: see the recent article (October, 1989) by Katye O. Sowell in this MAGAZINE. However the history seems to have fallen by the wayside along with the old geometry exhibit at the museum. The taxicab geometry is a simple way of introducing a non-Euclidean geometry to a general audience, and this indeed is done in the book *Taxicab Geometry* by Eugene F. Krause. Certainly Menger had in mind this educational value but he also saw its importance in a slightly different and more philosophical light.

Karl Menger was born in Vienna in 1902. He received his Ph.D. from the University of Vienna in 1924 under Hans Hahn. After serving as a docent at the University of Amsterdam, he returned to Vienna as a professor of geometry. At this time he joined, at Hahn's urging, a philosophical discussion group or circle now known as the Vienna Circle. The philosophy that developed from their meetings is logical positivism, or alternatively, as many of them preferred, logical empiricism.

One of the characteristics of this group and their philosophy was an intense dislike for traditional metaphysics. It is in this context that taxicab geometry becomes important, for in this geometry the equation $|\bar{x}| = 1$ defines a square. Thus we have a square circle which at that time was not a commonplace idea.

The metric concept was defined by Hermann Minkowski at the beginning of this century. Menger initiated the first systematic development of abstract distance geometry in 1928 with his four *Untersuchungen über allgemeine Metrik*. (See [1].) Indeed, as Menger puts it, "Square circles or round squares have haunted many diverse philosophical writers as the archetype of the impossible and the absurd; they were assigned a place near—or rather below—golden mountains, unicorns and mermaids. They have been discussed by Thomists, existentialists and linguistic philosophers as well as by Herbart, Bergson, Russell and many, many others." [3, p. 217] Thus one of Menger's interests in this geometry was to point out, like a true positivist, that traditional philosophy can discuss meaningless concepts at length. He concludes his article with the following: "Clearly these facts are of only limited importance for philosophers who, for their purposes, may replace the square circle by some other quasi-geometric *bête noire*. But they illustrate a new state of affairs. In the past, mathematicians have pleaded inability to associate meaning with certain metaphysical ideas. In dealing with square circles they, conversely, associate meaning with (and even give a practical interpretation to) the philosopher's paramount example of absurdity." [3, p. 219]

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On Coloring a Map

Guthrie shaded the map's every section,
Of Albion's fair isle with affection,
Yet his pencils grew duller,
Drawing regions to color,
So de Morgan he sought for direction.

A "quaternion" of color embraces,
Maps Hamilton fiercely retraces,
Oh conjecture so nice,
Four colors suffice,
Whether England or far-away places.

Heawood found the map theorem demanding,
And the steps in a proof e'er expanding,
With Möbius and Cayley,
He considered it daily,
Still the problem defied understanding.

Said Kempe, "my color connection,
Is a 'proof' which demands some perfection."
Greater fame it thus gained,
As the problem remained,
Till Appel and Haken's correction.

—RICHARD L. FRANCIS
Southeast Missouri State University

Note: The origin of the four-color map problem poses an unanswered question. Some attribute the remarkable conjecture to Francis Guthrie of the nineteenth century. His discovery stemmed from coloring a map of England. A succession of great mathematicians holds a prominent place in the continuing story.

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A Single Inequality Condition for the Existence of Many r-gons

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In [1] it was found that if $a_1, a_2, \ldots, a_n$ are positive numbers with $n \geqslant 3$, then a sufficient condition that every three of them are the lengths of sides of a triangle is given by

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^2 > (n-1)\left(a_1^4 + a_2^4 + \cdots + a_n^4\right).$$

It was left as an open problem to find a *single* inequality condition on the above n numbers which is both a necessary and sufficient condition that any r of them are the lengths of sides of an r-gon ($n \geqslant r \geqslant 3$).

If the a_i's were ordered, i.e., $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, it then follows easily that the desired inequality condition is

$$a_1 + a_2 + \cdots + a_{r-1} > a_n. \tag{1}$$

However, this really involves a total of n inequalities. We now show how to obtain a single inequality condition, albeit rather complicated, that is equivalent to the latter n inequalities.

Let C_i denote any one of the $\binom{n}{r}$ combinations of r terms $x_1, x_2, \ldots, x_r$ from the n terms $a_1, a_2, \ldots, a_n$. Then a necessary and sufficient condition that the x_i's are lengths of sides of an r-gon is

$$P_i \equiv \Pi(S - 2x_j) > 0,$$

where $S = x_1 + x_2 + \cdots + x_r$. It now follows that a necessary and sufficient condition that every combination of r terms from $a_1, a_2, \ldots, a_n$ are lengths of sides of an r-gon is simply

$$\min P_i > 0,$$

where i is over all $\binom{n}{r}$ combinations.

Finally, we show how to obtain an explicit formula for the latter inequality in terms of the absolute value function. Let

$$M_{r+1} = M(M_r, P_{r+1}), \qquad r = 1, 2, \ldots, \binom{n}{r} - 1,$$

where $M_1 = P_1$ and $M(a,b) = (|a+b| - |a-b|)/2$. Then, $M_2 = \min(P_1, P_2)$, $M_3 = \min(P_1, P_2, P_3), \ldots$. Thus our single inequality condition is

$$M_s > 0, \quad \text{where } s = \binom{n}{r}.$$

It is still an open problem whether or not there exists a single polynomial inequality that does the job. Most likely, there is not.

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1. M. S. Klamkin, Simultaneous triangle inequalities, this MAGAZINE 60 (1987), 236–237.

A Formula Yielding an Approximate Solution for Some Higher Degree Trinomial Equations

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This note will offer a formula that yields a close approximation of one root for a limited class of higher degree equations, those that can be expressed in the form, $10^N = 10^V X^A + 10^W X^B$, with N, V, and W any real numbers and with A and B both having the same sign and absolute values $\geqslant 1$. It yields accurate values for positive or negative A and B and large or small values of 10^N.

To begin, we pretend that $10^W X^B$ has a value of zero, so that $10^N = 10^V X^A$, and $X = 10^{(N-V)/A}$. If we retain this value for X and then acknowledge $10^W X^B$ as a positive quantity, we get $10^W X^B = 10^{B(N-V)/A+W}$, and $10^W X^B/10^V X^A = 10^{B(N-V)/A+W-N}$. We will call this ratio 10^C, so $10^N = 10^V X^A(1+10^C)$.

The factor $(1+10^C)$ by which 10^N became enlarged in the above process represents the magnitude of the error caused by our initially ignoring the $10^W X^B$ term. To find the correct value of X, we must divide the X derived from $10^N = 10^V X^A$ by some power of $(1+10^C)$ so that, when X is entered into the full equation that includes a nonzero $10^W X^B$ term, 10^N will no longer be enlarged. That power will be closer to $1/A$ or $1/B$ depending on which term, $10^V X^A$ or $10^W X^B$, respectively, is more influential. We will use the ratio 10^C to determine how far along the gap $(1/A - 1/B)$ we should move, starting at $1/B$. Our formula is thus

$$X \doteq \frac{10^{(N-V)/A}}{(1+10^C)^{1/B+(1/A-1/B)[1/(1+10^C)]}},$$

where $C = B(N-V)/A + W - N$.

Accuracy A comparison between the formula's solutions and the first value of X a computer finds yielding a 10^N differing with the original 10^N by less than 0.001% reveals that, whether A and B are negative or positive, the formula is least accurate when $1 \leqslant N \leqslant 2$. In this range, the formula's error can be as high as 10%, but as N is increased or decreased, the accuracy improves. For $3 \leqslant N \leqslant 4$ or $-2 \leqslant N \leqslant -1$, the maximum error is less than 1%, and for $6 \leqslant N \leqslant 7$ or $-5 \leqslant N \leqslant -4$, less than 0.1%.

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Transforms, Finite Fields, and Fast Multiplication

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Introduction The theme of a transform arises frequently. It entails transforming a problem into a setting where the new problem can be solved more easily and then applying the inverse transform to bring the solution back to the desired form. This is the general idea behind the fast Fourier transform (FFT) in a finite field for doing fast multiplication of large integers with absolute precision. Possible applications are to primality testing and cryptography where perfect accuracy is crucial.

Long ago in the seventeenth century, one highly effective use of a transform to make multiplication more tractable had been discovered, and was indispensable up to the computer age. In 1614 John Napier invented logarithms which transformed the multiplication problem into simple addition of exponents. Shortly after, Edward Gunter and William Oughtred built the powerful mechanical realization: the slide rule ([4, p. 247]).

The problem of multiplying big integers With the advent of the silicon chip, the celluloid slide rule became obselete. However, the problem of efficient multiplication lingered. It is now much more ambitious: given a computer capable of performing the operations $(+, -, *, \text{div})$ on the integers having up to l binary digits, how fast can it multiply two large n-digit numbers?

First, we need to state a decent method for measuring speed. Given a multiplication algorithm for large numbers, which invariably reduces to a bunch of l-digit size basic operations, one good method is to count the number of l-digit multiplications it takes to carry out the algorithm. As multiplications are slower than additions and occur more often in the algorithms we shall examine, they are a reasonable indication of speed.

Before discussing the fast Fourier transform, the classic grade school algorithm deserves a bit of analysis. When we multiply 123 by 456, we do this:

$$\begin{array}{r} 456 \\ \underline{123} \\ 1368 \\ 912 \\ \underline{456} \\ 56088 \end{array}$$

If we think of ourselves as computers that can multiply one-digit numbers, then the above takes $3^2 = 9$ multiplications. In general, to multiply two n-digit numbers requires n^2 multiplications. Extending the argument one step further, a computer capable of multiplying l-digit numbers would separate the n-digit numbers into $[n/l]+1$ chunks, and multiplying chunk by chunk requires $([n/l]+1)^2$ or $O(n^2)$ multiplications to perform the classic grade school algorithm.[1]

[1]A function $f(x)$ is $O(g(x))$ read "big oh of g," if $|f(x)| \leqslant Cg(x)$ for some constant C and all large x; it is used to indicate how fast a function grows.

In contrast, the fast Fourier transform will essentially allow us to multiply in $O(n \log n)$ multiplications!

Roots of the fast Fourier transform... In 1965 James Cooley and John Tukey published their famous paper [3], "An Algorithm for the Machine Calculation of Complex Fourier Series," which revolutionized that branch of mathematics. The method had been discovered forty years earlier by Runge and Konig, and independently by Stumpff, but in the precomputer days it was simpler to compute with the old $O(n^2)$ methods (for details, see [2]). It was a case of an idea coming before its time; only after the birth of the computer which can follow complicated instructions perfectly and rapidly did the fast Fourier transform realize its great potential. With so many applications in areas like spectral analysis, signal processing, and solution of differential equations, it was deemed as a major breakthrough ([1]).

...Transplantation to the world of algebra Although the fast Fourier transform was originally developed in an analytic setting of complex variables, in dealing with the concept of multiplication, an algebraic approach seems more appropriate. The following treatment is due to Lipson [5].

In the grade school algorithm, the integers are represented by the familiar positional notation. This is equivalent to polynomials to be evaluated at the base; for example, $456 = 4x^2 + 5x + 6$ at $x = 10$. *Therefore, if we can multiply polynomials quickly, then we can multiply large numbers quickly.* The fast Fourier transform provides a shortcut for multiplying polynomials.

What determines a polynomial? Obviously the coefficients do. More subtly, so do the evaluations of the polynomial at n distinct points where $n-1$ is the degree.

THEOREM 1 (Lagrange Interpolation Formula). *Let $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ be n elements in a field F, and $\beta_0, \beta_1, \dots, \beta_{n-1}$ be distinct in F. Then there exists a unique polynomial $f(x)$ in $F[x]$ of degree less than n such that $f(\beta_i) = \alpha_i$, $i = 0, 1, \dots, n-1$.*

Proof. The polynomial

$$f(x) = \sum_{i=0}^{n-1} \alpha_i (x-\beta_0) \cdots (x-\beta_{i-1})(x-\beta_{i+1}) \cdots (x-\beta_{n-1}) /$$

$$(\beta_i - \beta_0) \cdots (\beta_i - \beta_{i-1})(\beta_i - \beta_{i+1}) \cdots (\beta_i - \beta_{n-1})$$

is of degree less than n which works neatly. To establish uniqueness, suppose $g(x)$ in $F[x]$ is of degree less than n which also works. Then $g-f$ is a polynomial of degree less than n having n zeros since $g-f$ is zero at each distinct β_i. But in general, every non-trivial polynomial of degree d has at most d zeros in the field. Hence $g-f$ must be identically 0, implying $g=f$.

Let $a(x) = \sum_{i=0}^{n-1} a_i x^i$ and $b(x) = \sum_{i=0}^{n-1} b_i x^i$ be polynomials corresponding to $(a_{n-1} \cdots a_0)_B$ and $(b_{n-1} \cdots b_0)_B$, the two n-digit integers in base B to be multiplied. B is chosen to be small enough so that the computer can handle the a_i's, large enough to be efficient; B is typically a large integer less than 2^l. Hereafter, no confusion should arise if "multiplication of l-digit integers (or coefficients)" is simplified to just "multiplication." Now $c(x) = a(x)b(x)$ has degree less than $2n$. Let $N = 2n$, and ω be a primitive Nth root of unity (ω is a primitive Nth root if N is the least integer such that $\omega^N = 1$, for example, $e^{2\pi i/N}$ in the complex number system). The strategy will be:

A. FORWARD TRANSFORM

Evaluate $a(x)$ and $b(x)$ on $\{1, \omega, \omega^2, \ldots, \omega^{N-1}\}$, where ω is a primitive Nth root of unity, to obtain $\{a(1), a(\omega), \ldots, a(\omega^{N-1})\}$ and $\{b(1), b(\omega), \ldots, b(\omega^{N-1})\}$.

B. SOLVE IN NEW SETTING

Multiply pointwise $c(\omega^i) = a(\omega^i)b(\omega^i)$ to get $\{c(1), c(\omega), \ldots, c(\omega^{N-1})\}$. By the above theorem these points determine uniquely the polynomial $c(x) = a(x)b(x)$. Notice that we have *eliminated the need to deal with the cross terms* which arise when multiplying polynomials in the straightforward way.

C. INVERSE TRANSFORM

Interpolate the polynomial $c(x)$. End by rewriting it as $(c_{N-1} \cdots c_0)_B$.

Evaluation on $\{1, \omega, \omega^2, \ldots, \omega^{N-1}\}$ will be done by the procedure FFT to be described shortly. For interpolation, the Lagrange formula—while useful as a heuristic device—is far too clumsy for computing. Amazingly, interpolation in the inverse transform can be accomplished by basically the *same* procedure FFT used for evaluation in the forward transform.

There is also symmetry at a lower level. The set $W_N = \{1, \omega, \omega^2, \ldots, \omega^{N-1}\}$ possesses an important property of symmetry:

THEOREM 2. *Let $N = 2n$ be an even positive integer, ω an Nth root of unity, and $W_N = \{1, \omega, \omega^2, \ldots, \omega^{N-1}\}$. If ω^i is in W_N, then $-\omega^i = \omega^{n+i}$ is in W_N.*

Proof. $(\omega^n)^2 - 1 = 0$ implies that $\omega^n = 1$ or -1; but ω being a primitive $2n$th root of unity means that $\omega^n = -1$. Now $-\omega^i = (-1)\omega^i = \omega^n\omega^i = \omega^{n+i}$ lies in the cyclic group W_N.

Notice that when we square each element of W_N, we obtain the cyclic group $W_{N/2}$ of $N/2$ elements, where ω^2 is a primitive $N/2$th root of unity. Applying theorem 2 shows that $W_{N/2}$ has the above property of symmetry if $N/2$ is even. By choosing N to be a power of 2, then inductively every set $W_N, W_{N/2}, W_{N/4}, \ldots, W_2 = \{-1, 1\}$ would have the symmetry. Moreover, each set $W_{N/2^k}$ clearly contains $N/2^k$ distinct elements.

Forward transform unveiled The procedure FFT will exploit this property of symmetry to the fullest. In the forward transform—evaluating a polynomial $a(x) = \sum_{i=0}^{n-1} a_i x^i$ on the set W_N—we need only do the necessary multiplications for *at most half the points*. Make the substitution $s = x^2$, then

$$a(x) = \left(a_0 + a_2 s + a_4 s^2 + \cdots + a_{n-2} s^{n/2-1}\right)$$
$$+ x\left(a_1 + a_3 s + a_5 s^2 + \cdots + a_{n-1} s^{n/2-1}\right).$$

Once $a(x)$ is computed, $a(-x)$ can be obtained without any more multiplications because

$$a(-x) = \left(a_0 + a_2 s + a_4 s^2 + \cdots + a_{n-2} s^{n/2-1}\right)$$
$$- x\left(a_1 + a_3 s + a_5 s^2 + \cdots + a_{n-1} s^{n/2-1}\right)$$

as the -1 pops out of the odd terms. So far the number of multiplications is halved.

We can do even better. Each of the two polynomials $(a_0 + a_2 s + a_4 s^2 + \cdots + a_{n-2} s^{n/2-1})$ and $(a_1 + a_3 s + a_5 s^2 + \cdots + a_{n-1} s^{n/2-1})$ are just required to be evaluated on $W_{N/2}$ by the remarks of the last paragraph. The upshot is that having chosen the points $\{1, \omega, \omega^2, \ldots, \omega^{N-1}\}$ cleverly, evaluation on them requires substantially less multiplying than evaluation on N arbitrary distinct points.

For all subsequent evaluation purposes, given $f(x) = \sum_{i=0}^{N-1} f_i x^i$ to be evaluated on W_N, where N is a power of 2, use the following method.

PROCEDURE FFT

Step 1. Do if degree of $f(x)$ is 0.
Set answer as f_0.
End of procedure FFT.

Step 2. Do if degree of $f(x)$ is greater than 0.
a. Separate $f(x)$ into $p(x) = \sum_{i=0}^{N/2-1} f_{2i} x^i$ and $q(x) = \sum_{i=0}^{N/2-1} f_{2i+1} x^i$.
b. Recursively call procedure FFT to evaluate both $p(x)$ and $q(x)$, obtaining $p(\omega^{2k})$ and $q(\omega^{2k})$, for $k = 0, 1, \ldots, N/2 - 1$.
c. Set $f(\omega^k) = p(\omega^{2k}) + \omega^k q(\omega^{2k})$ and $f(\omega^{k+N/2}) = p(\omega^{2k}) - \omega^k q(\omega^{2k})$ for $k = 0, 1, \ldots, N/2 - 1$.

Now we estimate the speed.

THEOREM 3. *Procedure FFT requires* $(N/2)\log_2 N$ *multiplications to evaluate* $f(x) = \sum_{i=0}^{N-1} f_i x^i$ *on* W_N, *where N is a power of* 2.

Proof. Prove by induction. Let $M(N)$ be the number of multiplications for evaluation of $f(x)$ on W_N. The case $N = 1$ is trivial since Step 1 has no multiplications and $M(1) = 1/2 \log_2 1 = 0$. Assume the theorem to be true for all values less than N. Then the number of multiplications is the sum of the ones occurring in Step 2. Step 2.b takes $2M(N/2)$ and Step 2.c takes $(N/2)$. Hence

$$M(N) = 2M(N/2) + (N/2)$$

and by the inductive hypothesis,

$$M(N) = 2(N/4)\log_2(N/2) + (N/2)$$
$$M(N) = N/2(\log_2(N/2) + 1)$$
$$M(N) = N/2(\log_2 N).$$

The inverse transform—déjà vu Here, the problem is to interpolate the polynomial $c(x) = \sum c_i x^i$ given $\{c(1), c(\omega), \ldots, c(\omega^{N-1})\}$. Equivalently, we wish to find the c_i's in the following matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & & \omega^{N-1} \\ 1 & \omega^2 & (\omega^2)^2 & \cdots & (\omega^2)^{N-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{N-1} & (\omega^{N-1})^2 & \cdots & (\omega^{N-1})^{N-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} c(1) \\ c(\omega) \\ c(\omega^2) \\ \vdots \\ c(\omega^{N-1}) \end{bmatrix}.$$

The square matrix is called a Vandermonde matrix which is usually invertible. In our context, the inverse has an especially nice form when N^{-1} lives in the field.

$$N^{-1}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & (\omega^{-1})^2 & \cdots & (\omega^{-1})^{N-1} \\ 1 & \omega^{-2} & (\omega^{-2})^2 & \cdots & (\omega^{-2})^{N-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{-(N-1)} & (\omega^{-(N-1)})^2 & \cdots & (\omega^{-(N-1)})^{N-1} \end{bmatrix}$$

This is the inverse as readily checked. (Hint: think geometric series.) Hence

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix} = N^{-1}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \cdots & (\omega^{-1})^{N-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{-(N-1)} & \cdots & (\omega^{-(N-1)})^{N-1} \end{bmatrix}\begin{bmatrix} c(1) \\ c(\omega) \\ \vdots \\ c(\omega^{N-1}) \end{bmatrix}$$

Now the right-hand side is just an evaluation problem! The polynomial is $N^{-1}\Sigma_{i=0}^{N-1}c(\omega^i)x^i$; the points to be evaluated on are $\{1,\omega^{-1},(\omega^{-1})^2,\dots,(\omega^{-1})^{N-1}\}$. Since ω is a primitive Nth root of unity, so is ω^{-1}. Therefore we can simply use the procedure FFT to interpolate with, needing hardly any modification.

The multiplication algorithm Finally, putting all the pieces together according to the strategy outlined earlier, we devise an algorithm for multiplying large integers.

MULTIPLICATION ALGORITHM

Input: Integer $N = 2n$, a power of 2
Primitive Nth root of unity ω
Multiplicands $(a_{n-1} \cdots a_0)_B$, and $(b_{n-1} \cdots b_0)_B$
Output: Product $(c_{N-1} \cdots c_0)B$
Step 1. Evaluate $a(x) = \Sigma_{i=0}^{n-1}a_ix^i$ and $b(x) = \Sigma_{i=0}^{n-1}b_ix^i$ on $\{1,\omega,\omega^2,\dots,\omega^{N-1}\}$ by calling procedure FFT.
Step 2. Multiply pointwise to get $\{a(1)b(1), a(\omega)b(\omega),\dots,a(\omega^{N-1})b(\omega^{N-1})\}$.
Step 3. Interpolate $c(x) = \Sigma_{i=0}^{N-1}c_ix^i$ by evaluating $N^{-1}\Sigma_{i=0}^{N-1}a(\omega^i)b(\omega^i)x^i$ on $\{1,\omega^{-1},(\omega^{-1})^2,\dots,(\omega^{-1})^{N-1}\}$ by calling procedure FFT.
Step 4. Return $(c_{N-1} \cdots c_0)_B$.

THEOREM 4. *Let $N = 2n$ be an even integer, and F a field with N^{-1} and a primitive Nth root of unity. Then multiplying two n-digit integers can be accomplished in $O(n \log_2 n)$ multiplications.*

Proof. It suffices to show this for N a power of 2 because we are studying asymptotic behavior. To determine the speed, we count the number of multiplications occurring in the above algorithm. Step 1 uses $2(N/2)\log_2 N$ multiplications by theorem 3 in calling procedure FFT twice. Step 2 requires N multiplications, and Step 3 uses $(N/2)\log_2 N$. The total is

$$(3/2)N\log_2 N + N = 3n\log_2 2n + 2n = O(n\log_2 n).$$

Finite fields do have a lot to offer At this point, we have developed a fast multiplication algorithm over arbitrary fields in which exist N^{-1} and a primitive Nth root of unity. The question is which field to use. The fast Fourier transform's complex variables heritage together with the familiar primitive Nth root of unity $e^{2\pi i/N}$ make the field of complex numbers **C** the natural choice. Computational difficulties arise, however, since the typical computer lacks the routines to do complex arithmetic with absolute precision. Such routines would be prohibitive to write because the components of $e^{2\pi i/N}$ may be irrational. An elegant alternative was presented by Pollard in [6], who overcame the fixation on **C**. After all, **C** is not the only field that contains N^{-1} and a primitive Nth root of unity. We shall demonstrate that finite fields $\mathbf{Z}_p$, where p is prime and N divides $p-1$, have the desired properties. One pleasant consequence is that field multiplication uses the computer's built-in multiplication, plus a simple mod p reduction. Thus, the base B may be chosen to be a large prime, and multiplication is taken modulo B. (Technically, one also must worry about well-definedness.)

The key result describes the remarkably orderly structure of $\mathbf{Z}_p^x$, the multiplicative group of nonzero elements. It is true for all finite fields in general, and not much more difficult to prove.

THEOREM 5. *Let F be a finite field, F^x its multiplicative group of nonzero elements. Then F^x is cyclic.*

Proof. F is finite, so we may choose an element g having maximal order m, so $|F^x| \geqslant m$. F^x being a finite abelian group implies that every element of F^x has order dividing m, as readily checked. Hence, all the elements of F^x satisfy the equation $y^m - 1 = 0$. On the other hand, F is a field and has at most m zeros. Therefore, $|F^x| = m$, and g generates all of F^x.

From this theorem, we derive the criterion for a usable field $\mathbf{Z}_p$.

COROLLARY. *Let p be prime. If N divides $p-1$, then N^{-1} is in $\mathbf{Z}_p$ and $\mathbf{Z}_p$ has a primitive Nth root of unity.*

Proof. Obviously $N \leqslant p-1$, whence N is in $\mathbf{Z}_p$ and so must N^{-1}. $\mathbf{Z}_p^x$ is cyclic by the last theorem; let g be a generator, then $g^{(p-1)/N}$ is a primitive Nth root of unity.

Feasibility: a little handwaving goes a long way How hard is it to find these primes and their primitive roots? In practice, primes of the form $N = 2^r k + 1$ may be found in lists, e.g. see Riesel [7, pp. 381–83]. Their primitive roots are readily located by successively testing 2, 3, ... Philosophically, the multiplication algorithm over $\mathbf{Z}_p$ is feasible because given r, there exists an abundance of primes in the arithmetic sequence $2^r k + 1$ by appealing to an advanced result in number theory known as the Generalized Prime Number Theorem; and by another result in number theory, primitive roots make up more than 3 out of every π^2 elements in $\mathbf{Z}_p$ on the average ([5, p. 304]). Therefore, the primes and their primitive roots may be found in a reasonable amount of time.

Conclusion In the course of studying the fundamental concept of multiplication, many powerful ideas from different areas of mathematics come into play. The multiplication algorithm over a finite field is obviously not something that can be done effectively by hand, nor worth implementing on a computer for small numbers, so the grade school method still has its place. For large integers, such as those used in cryptography, using the fast Fourier transform is significantly faster and worth implementing.

Acknowledgements. Much thanks to two professors at UC San Diego: Dr. Lance Small for his inspiration and Dr. Ron Evans for his helpful comments. Also, the widely accessible computers at Stanford University were a great help.

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A Geometric Proof of Machin's Formula

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In the history of the computation of π, *Machin's formula*

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

has played an important rôle. Using Machin's formula and the Maclaurin series for the arctangent, William Shanks computed π to 707 places in 1873, a remarkable achievement for the time and an accomplishment which stood until 1946, when it was discovered that he had erred in the 528th place [2].

While the formula can be verified routinely using trigonometry, there is a simple constructive proof using only geometry and interpreting arctangents as angles. If we set $\theta = \arctan(1/5) = \arctan(30/150)$, then by use of similar triangles we find $2\theta = \arctan(60/144)$ [see FIGURE 1]. In a similar fashion, we find $4\theta = \arctan(119/120)$ [see FIGURE 2]. Extending the terminal side of 4θ and employing congruent triangles,

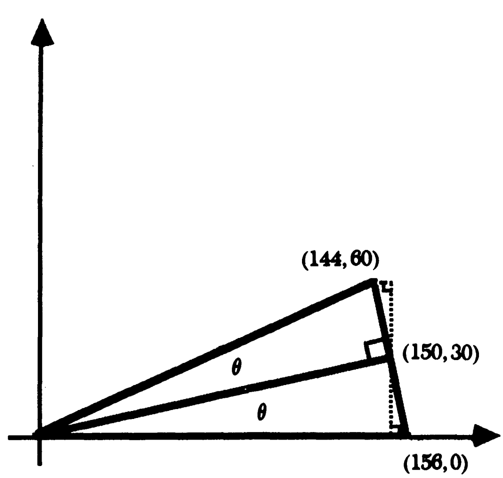

FIGURE 1

REFERENCES

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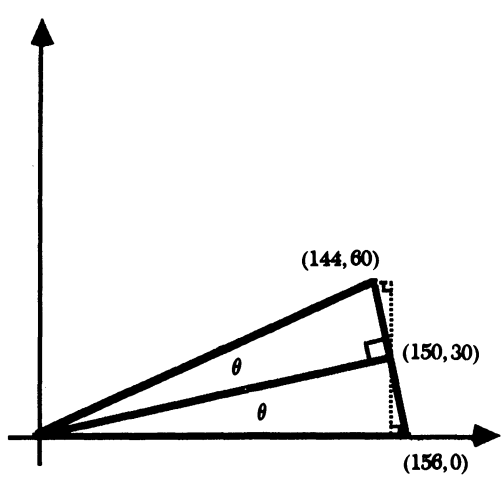

FIGURE 1

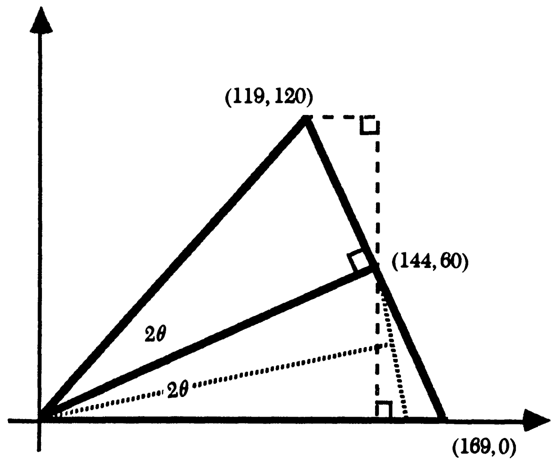

FIGURE 2

we obtain $4\theta = \pi/4 + \arctan(1/239)$ [see FIGURE 3], from which Machin's formula follows.

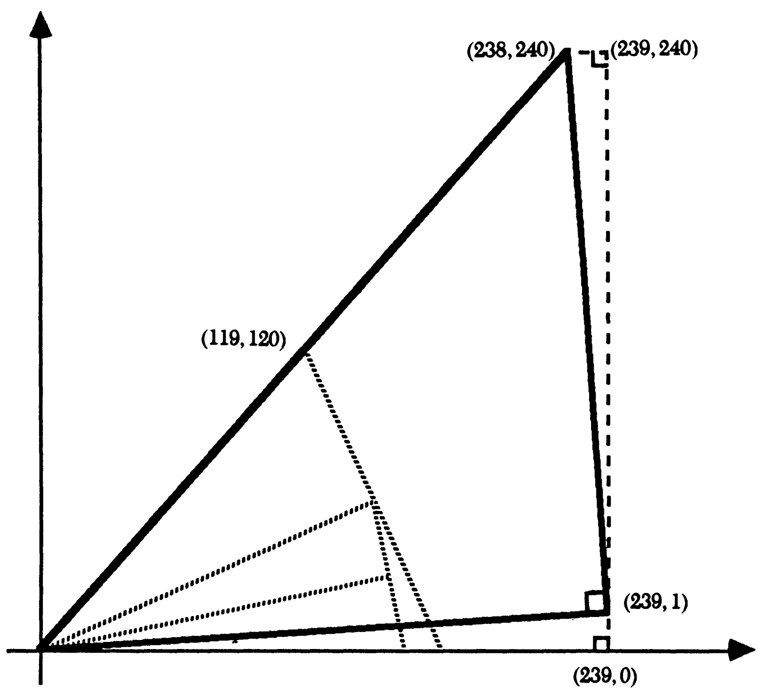

FIGURE 3

Expressions similar to Machin's formula have also been employed in the computation of π [1].

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The History of 2 + 2 = 5

HOUSTON EULER

First and above all he was a logician. At least thirty-five years of the half-century or so of his existence had been devoted exclusively to proving that two and two always equal four, except in unusual cases, where they equal three or five, as the case may be.

—Jacques Futrelle, "The Problem of Cell 13"

Most mathematicians are familiar with—or have at least seen references in the literature to—the equation $2+2=4$. However, the less well known equation $2+2=5$ also has a rich, complex history behind it. Like any other complex quantity, this history has a real part and an imaginary part; we shall deal exclusively with the latter here.

Many cultures, in their early mathematical development, discovered the equation $2+2=5$. For example, consider the Bolb tribe, descended from the Incas of South America. The Bolbs counted by tying knots in ropes. They quickly realized that when a 2-knot rope is put together with another 2-knot rope, a 5-knot rope results.

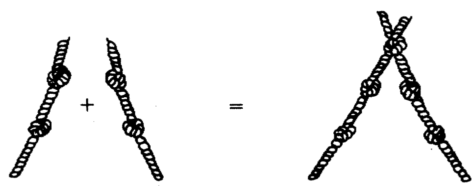

FIGURE 1
Incan-descent Bolb proof without words that $2 + 2 = 5$.

Recent findings indicate that the Pythagorean Brotherhood discovered a proof that $2+2=5$, but the proof never got written up. Contrary to what one might expect, the proof's nonappearance was not caused by a cover-up such as the Pythagoreans attempted with the irrationality of $\sqrt{2}$. Rather, they simply could not pay for the necessary scribe service. They had lost their grant money due to the protests of an oxen-rights activist who objected to the Brotherhood's method of celebrating the discovery of theorems. Thus it was that only the equation $2+2=4$ was used in Euclid's *Elements*, and nothing more was heard of $2+2=5$ for several centuries.

Around A.D. 1200 Leonardo of Pisa (Fibonacci) discovered that a few weeks after putting 2 male rabbits plus 2 female rabbits in the same cage, he ended up with considerably more than 4 rabbits. Fearing that too strong a challenge to the value 4 given in Euclid would meet with opposition, Leonardo conservatively stated, "2 + 2 is more like 5 than 4." Even this cautious rendition of his data was roundly condemned and earned Leonardo the nickname "Blockhead." By the way, his practice of underestimating the number of rabbits persisted; his celebrated model of rabbit populations had each birth consisting of only two babies, a gross underestimate if ever there was one.

Some 400 years later, the thread was picked up once more, this time by the French mathematicians. Descartes announced, "I think $2+2=5$; therefore it does." However, others objected that his argument was somewhat less than totally rigorous.

Apparently, Fermat had a more rigorous proof which was to appear as part of a book, but it and other material were cut by the editor so that the book could be printed with wider margins.

Between the fact that no definitive proof of $2+2=5$ was available and the excitement of the development of calculus, by 1700 mathematicians had again lost interest in the equation. In fact, the only known 18th-century reference to $2+2=5$ is due to the philosopher Bishop Berkeley who, on discovering it in an old manuscript, wryly commented, "Well, now I know where all the departed quantities went to—the right-hand side of this equation." That witticism so impressed California intellectuals that they named a university town after him.

But in the early to middle 1800's, $2+2$ began to take on great significance. Riemann developed an arithmetic in which $2+2=5$, paralleling the Euclidean $2+2=4$ arithmetic. Moreover, during this period Gauss produced an arithmetic in which $2+2=3$. Naturally, there ensued decades of great confusion as to the actual value of $2+2$. Because of changing opinions on this topic, Kempe's proof in 1880 of the 4-color theorem was deemed 11 years later to yield, instead, the 5-color theorem. Dedekind entered the debate with an article entitled "Was ist und was soll $2+2$?"

Frege thought he had settled the question while preparing a condensed version of his *Begriffsschrift*. This condensation, entitled *Die Kleine Begriffsschrift* (*The Short Schrift*), contained what he considered to be a definitive proof of $2+2=5$. But then Frege received a letter from Bertrand Russell, reminding him that in *Grundbeefen der Mathematik* Frege had proved that $2+2=4$. This contradiction so discouraged Frege that he abandoned mathematics altogether and went into university administration.

Faced with this profound and bewildering foundational question of the value of $2+2$, mathematicians followed the reasonable course of action: they just ignored the whole thing. And so everyone reverted to $2+2=4$ with next to nothing being done with its rival equation during the 20th century. There had been rumors that Bourbaki was planning to devote a volume to $2+2=5$ (the first forty pages taken up by the symbolic expression for the number five), but those rumors remained unconfirmed. Recently, though, there have been reported computer-assisted proofs that $2+2=5$, typically involving computers belonging to utility companies. Perhaps the 21st century will see yet another revival of this historic equation.

Covers by Linear Subspaces

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An interesting exercise of Herstein ([1] no. 21 p. 136) states the following:

> If V is a vector space over an infinite field F prove that V cannot be written as the set-theoretic union of a finite number of proper subspaces.

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Apparently, Fermat had a more rigorous proof which was to appear as part of a book, but it and other material were cut by the editor so that the book could be printed with wider margins.

Between the fact that no definitive proof of $2+2=5$ was available and the excitement of the development of calculus, by 1700 mathematicians had again lost interest in the equation. In fact, the only known 18th-century reference to $2+2=5$ is due to the philosopher Bishop Berkeley who, on discovering it in an old manuscript, wryly commented, "Well, now I know where all the departed quantities went to—the right-hand side of this equation." That witticism so impressed California intellectuals that they named a university town after him.

But in the early to middle 1800's, $2+2$ began to take on great significance. Riemann developed an arithmetic in which $2+2=5$, paralleling the Euclidean $2+2=4$ arithmetic. Moreover, during this period Gauss produced an arithmetic in which $2+2=3$. Naturally, there ensued decades of great confusion as to the actual value of $2+2$. Because of changing opinions on this topic, Kempe's proof in 1880 of the 4-color theorem was deemed 11 years later to yield, instead, the 5-color theorem. Dedekind entered the debate with an article entitled "Was ist und was soll $2+2$?"

Frege thought he had settled the question while preparing a condensed version of his *Begriffsschrift*. This condensation, entitled *Die Kleine Begriffsschrift* (*The Short Schrift*), contained what he considered to be a definitive proof of $2+2=5$. But then Frege received a letter from Bertrand Russell, reminding him that in *Grundbeefen der Mathematik* Frege had proved that $2+2=4$. This contradiction so discouraged Frege that he abandoned mathematics altogether and went into university administration.

Faced with this profound and bewildering foundational question of the value of $2+2$, mathematicians followed the reasonable course of action: they just ignored the whole thing. And so everyone reverted to $2+2=4$ with next to nothing being done with its rival equation during the 20th century. There had been rumors that Bourbaki was planning to devote a volume to $2+2=5$ (the first forty pages taken up by the symbolic expression for the number five), but those rumors remained unconfirmed. Recently, though, there have been reported computer-assisted proofs that $2+2=5$, typically involving computers belonging to utility companies. Perhaps the 21st century will see yet another revival of this historic equation.

Covers by Linear Subspaces

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An interesting exercise of Herstein ([1] no. 21 p. 136) states the following:

> If V is a vector space over an infinite field F prove that V cannot be written as the set-theoretic union of a finite number of proper subspaces.

Lord [3] suggests how the observation underlying this may be used to obtain a common algebraic complement for two linear subspaces of the same dimension in any finite-dimensional linear space, including those with finite fields. He shows, with an example of a vector space over $\mathbf{Z}/(2)$, that this fails for three linear subspaces. With the real or complex numbers as the scalar field and the use of the Baire Category Theorem, Lord extends the result for two linear subspaces to a countable family of linear subspaces.

This note also springs from the Herstein exercise but in a different direction. Our results show that a finite-dimensional linear space over an uncountable field cannot be covered by a countable family of proper linear subspaces. Of course, for real or complex numbers as the scalar field, this is a consequence of the Baire Category Theorem, yet our methods use only elementary algebra and cardinality arguments with no reference to topology. Moreover, Lord's result follows from these methods.

As any infinite-dimensional linear space has a cover by a countable family of hyperplanes, the Herstein exercise is no longer true, regardless of any restriction on the scalar field, when 'finite' is replaced by 'countable.' We state this known fact formally and prove it to contrast it better with the results of this paper.

PROPOSITION 1. *If E is an infinite-dimensional linear space over an arbitrary field, then E has a countable cover of proper linear subspaces.*

Proof. Since E is an infinite-dimensional linear space, it has an infinite Hamel basis, say $\boldsymbol{H}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of distinct elements of $\boldsymbol{H}$, and, for each n, let F_n be the linear span of $\boldsymbol{H}\setminus\{x_n\}$. Since each element x of E is a finite linear combination of elements of $\boldsymbol{H}$, there is an n such that x_n is not in this combination, and, consequently, x is in the corresponding F_n. Therefore the countable family of hyperplanes $\{F_n\}_{n=1}^{\infty}$ covers E.

Of course, if, in the above, E has a linear Baire space topology, some one of the hyperplanes (in fact, infinitely many according to the first corollary of the next result) must be dense in E and Baire itself (see Kelley, Namioka, et al. [2], problem B, p. 95). If the space E has finite dimension and an uncountable scalar field then no such countable cover exists as shown in Theorem 1, a consequence of the next result. The result itself shows that, regardless of the dimension, we regain some control on the covers when we restrict the cardinality of the scalar field.

PROPOSITION 2. *If the union of two countable families $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ of linear subspaces of linear space E with an uncountable scalar field is a cover of E, then one of them is a cover of E.*

Proof. Assume that neither $\boldsymbol{F}_1$ nor $\boldsymbol{F}_2$ covers E, and let x, y be elements of $E\setminus\bigcup\boldsymbol{F}_1$, $E\setminus\bigcup\boldsymbol{F}_2$, respectively. Since $\boldsymbol{F}_1\cup\boldsymbol{F}_2$ covers E, x and y are distinct. The line $L=\{rx+(1-r)y\colon r \text{ a scalar}\}$ passing through x and y is an uncountable subset of E. If an element F of $\boldsymbol{F}_k (k=1 \text{ or } 2)$ contains two distinct points of L, then, since F is a linear space, F contains L, and so F contains both x and y, a contradiction. Thus each element of $\boldsymbol{F}_k$ contains at most one point of L. Since $\boldsymbol{F}_1\cup\boldsymbol{F}_2$ is countable, it covers only a countable subset of the uncountable set L. Yet L is contained in $E=\bigcup(\boldsymbol{F}_1\cup\boldsymbol{F}_2)$, contradicting the assumption.

My attention was first drawn to this proposition and its relationship to the Herstein exercise by Stephen A. Saxon, and it comes in handy in analyzing certain covers of subspaces of products of linear topological spaces (see Todd and Saxon [4] p. 27 and Todd [5] p. 284).

There is a sense in which this last proof formalizes the *ad hoc* argument in 2-dimensions which Lord gives just before his main theorem. He observes that a countable family $\boldsymbol{F}$ of lines through the origin in 2-space does not cover the unit circle. In the above we obtain a line that cannot be contained in any member of $\boldsymbol{F}_1 \cup \boldsymbol{F}_2$ and, therefore, cannot be covered by such a countable collection $\boldsymbol{F}_1 \cup \boldsymbol{F}_2$.

In the context of an uncountable scalar field, the following establishes the Herstein exercise.

COROLLARY 1. *If E is a linear space with an uncountable scalar field and* $\boldsymbol{F}$ *is a countable cover of E by proper linear subspaces of E, then each cofinite subcollection of* $\boldsymbol{F}$ *covers E.*

Proof. Let F be in $\boldsymbol{F}$. Since F is a proper linear subspace of E, $\{F\}$ is not a cover. By the proposition, $\boldsymbol{F}\setminus\{F\}$ is a cover. Finite mathematical induction completes the proof.

As noted, the following is much in contrast to the first proposition. For the real or complex numbers as the scalar field, it is a result of the Baire Category Theorem. Here we treat it as a simple consequence of Proposition 2.

THEOREM 1. *If E is a finite-dimensional linear space over an uncountable field, then E has no countable cover by proper linear subspaces.*

Proof. For any countable cover $\boldsymbol{F}$ of E consisting of linear subspaces of E, observe that the family of those elements of $\boldsymbol{F}$ that do not contain a given element x of E does not cover $\{x\}$ so it can not cover E. By Proposition 2, its complement in $\boldsymbol{F}$, $\{F \in \boldsymbol{F} : x \in F\}$, does. For $\dim E = 0$, E has no proper linear subspaces, so let $n = \dim E > 0$ and $x_1, x_2, \dots, x_n$ be a basis for E. Suppose $\boldsymbol{F}_0$ is any countable cover of E consisting of linear subspaces of E. From the observation, $\boldsymbol{F}_1 = \{F \in \boldsymbol{F}_0 : x_1 \in F\}$ is a countable cover of E by linear subspaces of E. Similarly $\boldsymbol{F}_2 = \{F \in \boldsymbol{F}_1 \colon x_2 \in F\}$ is such a cover. By the choice of $\boldsymbol{F}_1$, each member of $\boldsymbol{F}_2$ contains x_1 and x_2. Finite induction gives, for $0 < k \leq n$, $\boldsymbol{F}_k = \{F \in \boldsymbol{F}_{k-1} : x_k \in F\} \subset \boldsymbol{F}_{k-1}$ is a cover of E by linear subspaces of E and each member of $\boldsymbol{F}_k$ contains $x_1, x_2, \dots, x_k$. Now for $k = n$ each member of $\boldsymbol{F}_n = \boldsymbol{F}_k$ contains $x_1, x_2, \dots, x_n$, a basis for E, so that $\boldsymbol{F}_n = \{E\} \subset \boldsymbol{F}_0$. Therefore no such countable cover consists only of proper linear subspaces of E.

Now Lord's theorem follows from this; we use his induction argument given for the case of two subspaces and restate the theorem for any uncountable scalar field.

THEOREM 2 (Lord). *Let* $\boldsymbol{F}$ *be a countable family of linear subspaces of a finite-dimensional linear space E with an uncountable scalar field. If each member of* $\boldsymbol{F}$ *has the same dimension k, then there is a linear subspace G of E such that G is an algebraic complement of each member of* $\boldsymbol{F}$.

Proof. If k is the dimension n of E (or is 0) then use $\{0\}$ (or E) for G. Otherwise $\boldsymbol{F}$ consists of proper linear subspaces of E and so does not cover E by Theorem 1. Let v_1 be an element of $E \setminus \bigcup \boldsymbol{F}$, and consider the countable family $\boldsymbol{F}_1 = \{F + \operatorname{span}(\{v_1\}) \colon F \in \boldsymbol{F}\}$ which consists of linear subspaces of E, each of dimension $k+1$. If $k + 1 < n = \dim E$, there is a v_2 in $E \setminus \bigcup \boldsymbol{F}_1$ so that $\boldsymbol{F}_2 = \{F + \operatorname{span}(\{v_2\}) \colon F \in \boldsymbol{F}_1\}$ is a countable family of linear subspaces of E each of dimension $k+2$. Since E has finite dimension, finite induction provides us with $v_1, v_2, \dots, v_m$ where $k + m = n = \dim E$, and for each F in $\boldsymbol{F}$, we have $F + \operatorname{span}(\{v_1, v_2, \dots, v_m\}) = E$, so that $G = \operatorname{span}(\{v_1, v_2, \dots, v_m\})$ is the required common algebraic complement.

We conclude by confirming directly, in terms of cardinality, Lord's observation that there are many common algebraic complements. (The author is grateful for the suggestions on the following by a particularly observant and public spirited referee.)

THEOREM 3. *Assume the hypotheses of the previous theorem, and let* $\mathbf{G}$ *be the collection of all G such that G is an algebraic complement in E of each member of* $\mathbf{F}$. *If* k, *the common dimension of the members of* $\mathbf{F}$, *is strictly between* 0 *and the dimension* n *of* E, *then* $\mathbf{G}$ *is uncountable.*

Proof. Since the dimensions k and $m = n - k$ of members of $\mathbf{F}$ and $\mathbf{G}$, respectively, are strictly less than $n = \dim(E)$, the members of $\mathbf{F} \cup \mathbf{G}$ are proper linear subspaces of E. If $\mathbf{G}$ is countable, then $\mathbf{F} \cup \mathbf{G}$ is countable, so there is a v in $E \setminus \bigcup(\mathbf{F} \cup \mathbf{G})$. As $\mathbf{F}_0 = \{F + \text{span}(\{v\}): F \in \mathbf{F}\}$ satisfies the hypotheses of Theorem 2, there is a linear subspace H of E that is an algebraic complement to each member of $\mathbf{F}_0$. Now $H + \text{span}(\{v\})$ is an algebraic complement to each member of $\mathbf{F}$, so $\text{span}(\{v\}) + H \in \mathbf{G}$. Now $v \in \bigcup \mathbf{G}$ which is a contradiction of the choice of v, and so $\mathbf{G}$ is uncountable.

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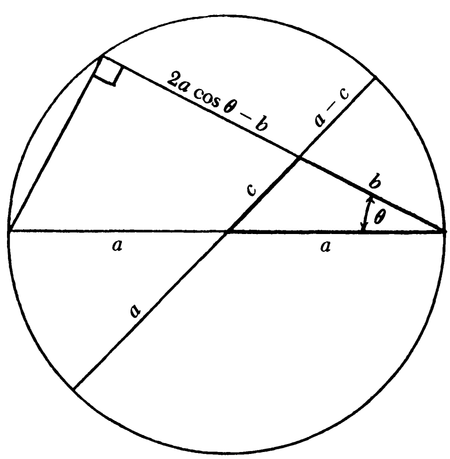
Proof without Words: The Law of Cosines



$$(2a\cos\theta - b)b = (a - c)(c + a)$$
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

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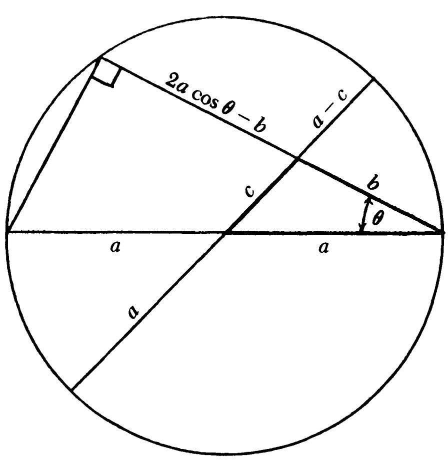

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Application of Cubic Splines to Contour Plotting

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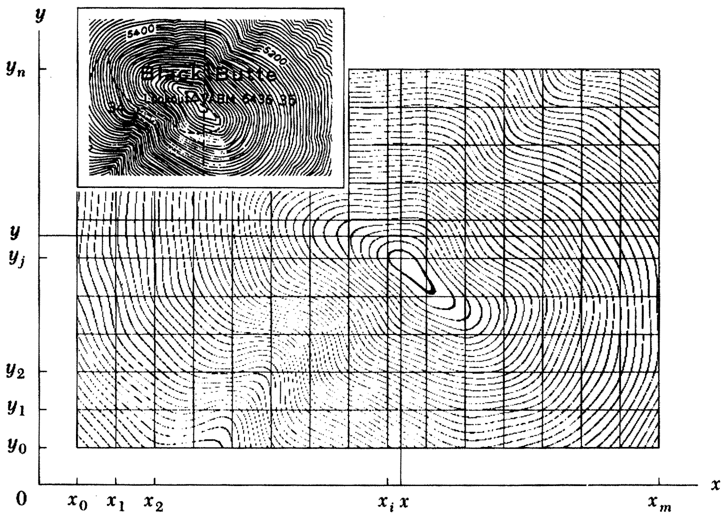

FIGURE 1
Contour map of Black Butte, Oregon, plotted at the Willamette University Computer Center. A USGS quad map is inset for comparison.

A contour map is a diagram used to represent topography of a land surface by contour curves of equal elevation. This geographical convention can be extended to analyze diverse objects such as an atmospheric pressure field, a stress field in a steel plate, and the variation in the density of debris at an archaeological site [3].

Consider a rectangle R partitioned into $m \times n$ rectangular panels of equal size with a regular grid of points (x_i, y_j) for $i = 0, 1, \dots, m$, $j = 0, 1, \dots, n$, and a real function f on R whose values at the grid points

$$z_{i,j} = f(x_i, y_j)$$

are known. In this paper, we are concerned with a construction of a continuous real function S on R that is continuously differentiable on the interior of R and whose values agree with the values of f at the grid points. Thus, S is an interpolating function of two variables whose graph is a smooth surface over the rectangle R containing all the data points $(x_i, y_j, z_{i,j})$. The implicit function theorem guarantees that such a function S has smooth contour curves, which can be plotted easily by a simple computer graphics device such as a dot matrix printer.

Various methods of contour plotting have been developed ([2], [5], [6]). The simplest method is given by a panelwise linear interpolation ([2]) which yields contour curves that are continuous but not smooth. By using a finer grid, these curves can be made visually smooth but a finer grid means a greater number of data points that must be available first. In [6] we find a method for smooth contour curves, which makes use of triangular panels instead of rectangular ones and a panelwise quadratic function with continuous differentiability across the boundaries of the panels. The method yields an impressive contour map with a relatively coarse grid but as a trade-off it requires gradient vectors of f as well as the values at the grid points.

Our object is to construct an interpolating function S with the aforementioned smoothness property, using cubic splines and the value-only data at the grid points. As we start with value-only data, cubic spline means natural cubic spline throughout this article. At the outset, we have presented a topographical map of Black Butte in Oregon, a mountain famous among Oregon hikers for its smooth contours. This map consists of contour curves of S interpolating to the altitudes measured at the grid points.

Given (x, y) in R, we define the value $S(x, y)$ using $n+2$ cubic splines as follows. Firstly, for each $j = 0, 1, \ldots, n$, let H_j be the cubic spline on the interval $[x_0, x_m]$ that agrees with the data values $z_{i,j} = f(x_i, y_j)$, $i = 0, 1, \ldots, m$. Thus, the restriction of H_j to the subinterval $[x_i, x_{i+1}]$ is the cubic function

$$H_{i,j}(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \tag{1}$$

for some a_i, b_i, c_i, and d_i satisfying the continuity property

$$H_{i+1,j}^{(r)}(x_{i+1}) = H_{i,j}^{(r)}(x_{i+1}) \tag{2}$$

for each $i = 0, 1, \ldots, m-2$, and $r = 0, 1, 2$. In addition to (1) and (2), H_j satisfies the free boundary condition

$$H_j''(x_0) = H_j''(x_m) = 0 \tag{3}$$

as well as

$$H_j(x_i) = f(x_i, y_j) = z_{i,j} \tag{4}$$

for each $i = 0, 1, \ldots, m$. We may refer to H_j as a horizontal spline interpolant for f as it interpolates to the data values $z_{i,j}$ over the jth horizontal grid line of R (see FIGURE 1). Using the straightforward algebraic process similar to that of [1], we obtain the following equations that determine the coefficients a_i, b_i, c_i, and d_i:

$$a_i = z_{i,j} \tag{5}$$

$$b_i = \frac{a_{i+1} - a_i}{h} - \frac{h}{3}(2c_i + c_{i+1}) \tag{6}$$

$$d_i = \frac{c_{i+1} - c_i}{3h} \tag{7}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & \ldots & 0 & 0 & 0 \\ & & & & & \ldots & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \\ c_{m-1} \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ (z_{0,j} - 2z_{1,j} + z_{2,j})g \\ (z_{i,j} - 2z_{2,j} + z_{3,j})g \\ \ldots \\ (z_{m-2,j} - 2z_{m-1,j} + z_{m,j})g \\ 0 \end{bmatrix}, \tag{8}$$

where $h = x_{i+1} - x_i$ and $g = 3/h^2$. Note that the tridiagonal linear system (8) can be solved very efficiently for c_i, $i = 0, 1, \ldots, m$, by a computer. Secondly, we define, using formulas similar to (1)–(4), the vertical spline V_x which interpolates to the values $H_j(x)$, $j = 0, 1, \ldots, n$, over the vertical line in R through the point (x, y) (see FIGURE 1). To avoid the redundancy, we shall write only the equations corresponding to (1) and (5)–(8):

$$V_{x,j}(y) = \alpha_j + \beta_j(y - y_j) + \gamma_j(y - y_j)^2 + \delta_j(y - y_j)^3 \tag{9}$$

$$\alpha_j = H_j(x) \tag{10}$$

$$\beta_j = \frac{\alpha_{j+1} - \alpha_j}{k} - \frac{k}{3}(2\gamma_j + \gamma_{j+1}) \tag{11}$$

$$\delta_j = \frac{\alpha_{j+1} - \alpha_j}{3k} \tag{12}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & \ldots & 0 & 0 & 0 \\ & & & & & \ldots & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \ldots \\ \gamma_{n-1} \\ \gamma_n \end{bmatrix} = \begin{bmatrix} 0 \\ (H_0(x) - 2H_1(x) + H_2(x))p \\ (H_1(x) - 2H_2(x) + H_3(x))p \\ \ldots \\ (H_{m-2}(x) - 2H_{m-1}(x) + H_m(x))p \\ 0 \end{bmatrix}, \tag{13}$$

where $V_{x,j} = V_x | [y_j, y_{j+1}]$, $k = y_{j+1} - y_j$, and $p = 3/k^2$. Observe from (9)–(13) that all of α_j, β_j, γ_j and δ_j are linear functions of $H_0(x), H_1(x), \ldots, H_m(x)$. Therefore, by (2), they are continuous and have continuous derivatives of order 1 and 2 with respect to x. It is then obvious from (9) and the definition of V_x that $V_x(y)$ is continuous and has continuous partial derivatives with respect to both x and y. We finally define

$$S(x, y) = V_x(y).$$

We have just proved that S is continuous on R and continuously differentiable on the interior of R.

REMARK. The basic technique in our construction, "horizontal interpolations" followed by "vertical interpolations," can be used with any other standard interpolating functions for surface fitting. The use of higher order Lagrange polynomials, however, results in extra peaks and valleys due to their inherent edge-effects. Reference [4] mentions a surface fitting method which uses a double summation of products of B-splines (B for "Bell-shaped"). It can be proved that our cubic spline construction and the B-spline method yield the same function.

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Inequalities of the Form $f(g(x)) \geqslant f(x)$

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Consider a problem in trigonometry [1]: To prove that for all x in the interval $[0, \pi]$,

$$(\sin x)(1+\cos x) \leqslant \left[\sin\left[\frac{x+\pi}{4}\right]\right]\left[1+\cos\left[\frac{x+\pi}{4}\right]\right]. \tag{1}$$

It is possible to solve this problem by manipulating both sides using trigonometric identities, but this method leads to complicated expressions which yield little insight into the problem. A different approach is to reformulate the problem in functional terms: It is the same as proving that

$$F(G(x)) \geqslant F(x), \tag{2}$$

where $F(x) = (\sin x)(1+\cos x)$ and $G(x) = (x+\pi)/4$.

In this note, a simple method is given for constructing functions G such that (2) holds for a given function F and all x in a given closed interval $[a, b]$. The method will then be illustrated in two applications. The related functional inequality $F(G(x)) \geqslant G(F(x))$ is considered in [2].

Let F and G denote continuous functions on a finite closed interval $[a, b]$. If the following general conditions are satisfied, then (2) holds for all x in $[a, b]$:

C1: Interval $[a, b]$ is partitioned into N closed subintervals $\{I_n,\ 1 \leqslant n \leqslant N\}$ such that for each n, F is monotonic on I_n.
C2: For each n, G maps I_n into I_n.
C3: In each subinterval where F increases, $G(x) \geqslant x$. In each subinterval where F decreases, $G(x) \leqslant x$.

Graphically, C2 and C3 imply that for each $n \leqslant N$, the graph of G lies within the square $I_n \times I_n$, above or below the line $y = x$ according to whether F is increasing or decreasing in I_n. This is illustrated in FIGURE 1.

The trigonometric inequality above can be proven using these ideas. As above, let $F(x) = (\sin x)(1+\cos x)$ and $G(x) = (x+\pi)/4$. F is increasing in $[0, \pi/3]$ and decreasing in $[\pi/3, \pi]$. Each linear function $G(x) = Ax + B$ satisfies conditions C2 and C3 provided that $G(\pi/3) = \pi/3$ and $0 \leqslant A \leqslant 1$; the G given in the statement of the problem is a member of a continuous family of linear functions for which the inequality holds.

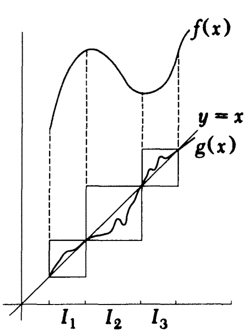

FIGURE 1

The conditions stated above can be used to prove an interesting theorem. Define $P(x) = G(x) - x$. Then C3 implies that sgn $P(x) =$ sgn $F'(x)$ for all x in (a, b). This suggests that there should exist a 'small' positive real constant k such that $G(x) = x + kF'(x)$ would satisfy conditions C2 and C3. However, if $F'(a) < 0$, then this would imply that $G(a) = a + kF'(a) < a$, contradicting condition C2 requiring that G map $[a, b]$ into itself. The same problem would arise if $F'(b)$ were positive. By introducing the non-negative multiplier $(x - a)(b - x)$, this problem is avoided.

THEOREM. *Let F be a twice-differentiable function with a finite number of local extrema in $[a, b]$. Then there exists a positive real constant k such that for all x in $[a, b]$,*

$$F(x) \leqslant F[x + k(x-a)(b-x)].$$

Proof. The finite set of local extrema of F furnishes a subdivision of $[a, b]$ into intervals on which F is monotonic. Consider one such interval $I = [c, d]$ on which F increases. To satisfy C2 and C3, it is sufficient to determine a positive real constant k such that the following inequality is satisfied in I:

$$c \leqslant x \leqslant x + k(x-a)(b-x)F'(x) \leqslant d.$$

The two leftmost inequalities are obvious. The last inequality holds iff

$$k(x-a)(b-x)F'(x) \leqslant d - x.$$

Case 1: $d = b$. The inequality will hold when

$$k(x-a)F'(x) \leqslant 1.$$

The existence of F'' implies continuity and boundedness of F'; let M be the maximum absolute value of F' in $[a, b]$.

$$k(x-a)F'(x) \leqslant k(b-a)M; \quad \text{choose } k = \frac{1}{M(b-a)}.$$

Case 2: $d < b$. Then $F'(d) = 0$ and

$$k(x-a)(b-x)[F'(x) - F'(d)] \leqslant d - x$$

$$\Leftrightarrow \qquad k(x-a)(b-x)[(F'(x) - F'(d))/(d-x)] \leqslant 1.$$

Since the difference quotient approaches $-F'(d)$ as x approaches d, there exists a number t in (c, d) such that the quotient is bounded above in (t, d). Therefore, the absolute value of the difference quotient in $[c, d]$ is bounded above by some positive number M. Also, $(x-a)(x-b) < (b-a)^2$. Take

$$k = \frac{1}{M(b-a)^2}$$

and the inequality is satisfied. A similar argument applies if F decreases on I. Thus the above shows how to chose k in any specific subinterval I. Taking the minimum value of k over all the subintervals of the partition, one obtains a value of k valid for the entire interval $[a, b]$. Q.E.D.

The same argument can be used to establish the existence of a positive real constant h such that for all x in $[a, b]$,

$$F(x) \geqslant F[x - h(x-a)(b-x)F'(x)].$$

If $F'(a) \geqslant 0$, then the multiplier $(x-a)$ in the statement of the theorem is superfluous, and if $F'(b) \leqslant 0$, then $(b-x)$ is superfluous. For example, taking $F(x) = \sin x$ and $[a, b] = [0, \pi]$, we find that G can be of the form $G(x) = x + kF'(x)$. Applying to this pair of functions the argument used to prove the theorem, we find that for $0 \leqslant x \leqslant \pi$,

$$0 \leqslant k \leqslant 1 \Rightarrow \sin(x + k(\cos x)) \geqslant \sin x.$$

Setting $x = \pi/2 - y$, we obtain the equivalent inequality

$$\cos(y - k(\sin y)) \geqslant \cos y, \quad -\pi/2 \leqslant y \leqslant \pi/2.$$

Other inequalities of similar form may be written down by varying the function or the interval.

REFERENCES

1. *The Arbelos*, March 1987, p. 11, problem 20.
2. Boas, R. P., Inequalities for a Collection, this MAGAZINE, 52 (Jan. 1979), 29–31.

Poetry Analysis

RICHARD MCDERMOT
Allegheny College

DENNIS MCDERMOT
University of Tennessee

While watching the classic film, *Singing in the Rain*, recently, we were intrigued by the curious behavior of the protagonist in the song whose lyrics are

Moses supposes his toeses are roses,
But Moses supposes erroneously.
Moses he knowses his toeses aren't roses
As Moses supposes his toeses to be.

Upon analysis, we concluded that there are only two[1] reasonable explanations for the thought patterns attributed to Moses in this poem.

a) Moses is suffering from an obsessive-compulsive disorder, which compels him to irrational beliefs even though he is aware of their irrationality, or[2]
b) Moses is a mathematician who is attempting to prove by contradiction that his toeses are not roses.

---

[1]After briefly considering a multiple personality as an additional possibility, we decided that, if this were the case, the third line would be

Abraham knowses his toeses aren't roses.

[2]As usual, this is the inclusive 'or.'

$$F(x) \geqslant F[x - h(x-a)(b-x)F'(x)].$$

If $F'(a) \geqslant 0$, then the multiplier $(x-a)$ in the statement of the theorem is superfluous, and if $F'(b) \leqslant 0$, then $(b-x)$ is superfluous. For example, taking $F(x) = \sin x$ and $[a, b] = [0, \pi]$, we find that G can be of the form $G(x) = x + kF'(x)$. Applying to this pair of functions the argument used to prove the theorem, we find that for $0 \leqslant x \leqslant \pi$,

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[1]After briefly considering a multiple personality as an additional possibility, we decided that, if this were the case, the third line would be

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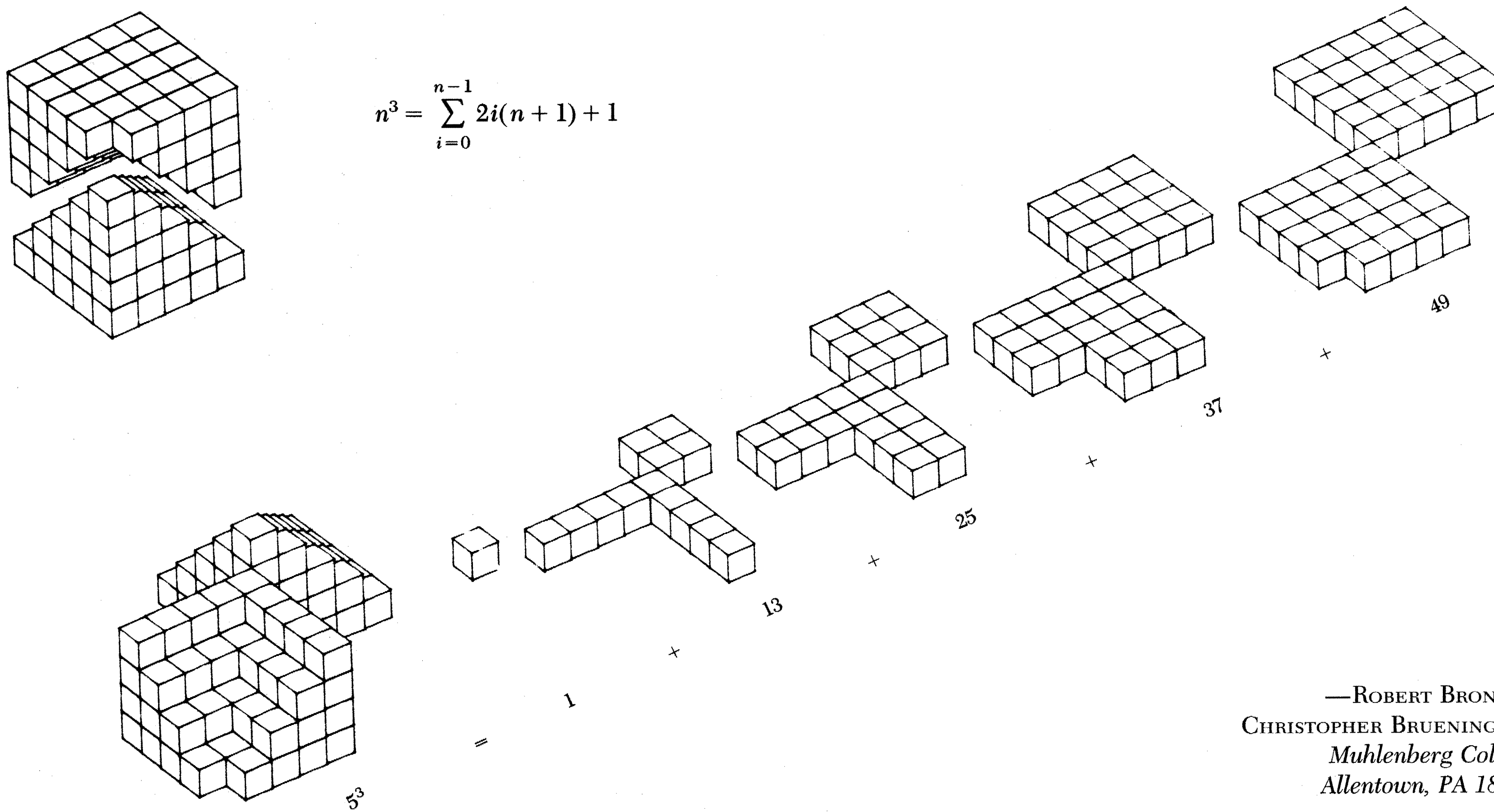
[2]As usual, this is the inclusive 'or.'

Proof without Words: The Cube as an Arithmetic Sum

$$n^3 = \sum_{i=0}^{n-1} 2i(n+1) + 1$$



—ROBERT BRONSON
CHRISTOPHER BRUENINGSEN
Muhlenberg College
Allentown, PA 18104

PROBLEMS

LOREN C. LARSON, *editor*
St. Olaf College

GEORGE GILBERT, *associate editor*
Texas Christian University

Proposals

To be considered for publication, solutions should be received by May 1, 1991

1358. *Proposed by Peter Ross, Santa Clara University, Santa Clara, California.*

Which positive integers are representable in the form $\binom{k}{2} + kn$, $k > 1$, $n \geqslant 1$?

1359. *Proposed by Frank Schmidt, Bryn Mawr College, Bryn Mawr, Pennsylvania.*

Let $\mathbf{A} = (a_{ij})$ be a real $n \times n$ positive definite matrix.

(a) Show that $\prod_{i=1}^{n} a_{ii} > \prod_{i=1}^{n} a_{i\sigma(i)}$ for all $\sigma \in S_n$, $\sigma \neq$ identity.

(b) Show that $\sum_{i=1}^{n} a_{ii} > \sum_{i=1}^{n} a_{i\sigma(i)}$ for all $\sigma \in S_n$, $\sigma \neq$ identity.

1360. *Proposed by Paul R. Scott, University of Adelaide, Adelaide, Australia.*

Let A be a bounded convex set in the plane and B a set congruent to it. Given that $A \cap B$ is a centrally symmetric set for all positions of A and B, must A be a circular disc?

1361. *Proposed by Mark Krusemeyer, Carleton College, Northfield, Minnesota.*

Does there exist a differentiable function f, defined for all real numbers x, such that $f(f(x)) = e^x$ for all x? If so, exhibit such a function; if not, show why not.

1362. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

If (a_i, b_i, c_i) are the sides, R_i the circumradii, r_i the inradii, and s_i the semi-perimeters of n triangles ($i = 1, 2, 3, \ldots, n$) respectively, show that

ASSISTANT EDITORS: CLIFTON CORZAT and THEODORE VESSEY, *St. Olaf College. We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren Larson, Department of Mathematics, St. Olaf College, Northfield, MN 55057.

$$3\sum\left(\prod a_i^{-1/n}+\prod b_i^{-1/n}+\prod c_i^{-1/n}\right)\leqslant\prod\left(\frac{s_i}{r_iR_i}\right)^{1/n}$$
$$\leqslant 2^n\prod\left(a_i^{-1/n}+b_i^{-1/n}+c_i^{-1/n}\right),$$

where the sums and products are over $i=1$ to n.

Quickies

Answers to the Quickies are on page 357.

Q769. *Proposed by Norman Schaumberger, Bronx Community College, Bronx, New York.*

If $0<\theta<\pi/2$, then $\sin 2\theta\geqslant(\tan\theta)^{\cos 2\theta}$

Q770. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Canada.*

Determine the minimum value of $x^2+y^2+z^2$ given that

$$xyz-x-y-z=2 \quad\text{and}\quad x,y,z\geqslant 0.$$

Q771. *Proposed by Lennart Råde, Chalmers Technical University and Göteborgs University, Göteborg, Sweden.*

Let M be the largest number in a finite set of real numbers $\{x_1,x_2,\dots,x_n\}$ and let $m_{i_1i_2\cdots i_k}$ be the smallest number in the subset $\{x_{i_1},x_{i_2},\dots,x_{i_k}\}$. Prove that

$$M=\sum_i m_i-\sum_{i<j}m_{ij}+\sum_{i<j<k}m_{ijk}+\cdots+(-1)^{n+1}m_{12\dots n}.$$

Solutions

Partial sum permutations **December 1989**

1332. *Proposed by Bernardo Recamán, Instituto Alberto Merani, Colombia, and Michael Zielinski, St. John's University, Collegeville, Minnesota.*

The permutation $\{8,5,2,7,4,1,6,3\}$ has the property that its partial sums, taken left to right modulo 8, also form a permutation, namely, $\{8,5,7,6,2,3,1,4\}$. For each n, let P_n denote the set of all permutations of $\{1,2,\dots,n\}$ with this property. That is, $\mathbf{x}=(x_1,x_2,\dots,x_n)\in P_n$ if and only if $\mathbf{x}$ is a permutation of $\{1,2,\dots,n\}$ and $s(\mathbf{x})\equiv(s_1,s_2,\dots,s_n)$ is also a permutation where $s_i\equiv x_1+x_2+\cdots+x_i \pmod n$, $1\leqslant s_i\leqslant n$.

(a) Prove that $P_n\neq\varnothing$ if and only if n is even.

(b) Prove that if $\mathbf{x}\in P_n$, then $s(\mathbf{x})\notin P_n$.

Solution by Helen M. Marston, retired, Douglass College, Princeton, New Jersey.

Let $\mathbf{x}=(x_1,x_2,\dots,x_n)$ be a permutation of $\{1,2,\dots,n\}$, and let all congruences be modulo n. The result of (a) is not true for $n=1$ (take $P_1=\{(1)\}$), so suppose that $n>1$. Furthermore, (b) is not true for $n=2$, because in this case, $P_2=\{(2,1)\}$, and this contradicts (b). So, suppose that $n>1$ for (a) and $n>2$ and even for (b).

Note that if $\mathbf{x}\in P_n$, then $x_1=n$. For suppose that $x_j=n$ for $j>1$. Then $s_{j-1}\equiv\sum_{i=1}^{j-1}x_i\equiv\sum_{i=1}^{j}x_i\equiv s_j$, a contradiction.

(a) If n is odd, then $s_n = \sum_{i=1}^n x_i \equiv \sum_{i=1}^n i \equiv n(n+1)/2 \equiv n$. Hence $s_1 = s_n$, so $s(\mathbf{x})$ is not a permutation. Therefore, if $\mathbf{x} \in P_n$, then n is even.

If n is even, then $\mathbf{x} = (n, 1, -2, 3, -4, \dots, n-1) \in P_n$. Hence, if n is even, then $P_n \neq \emptyset$.

(b) We have $s_n \equiv \sum_{i=1}^n i = n(n+1)/2 \equiv n/2$. If $s(\mathbf{x}) \in P_n$, then $\sum_{i=1}^n s_i \equiv n/2$ and $s_n = n/2$, so $\sum_{i=1}^{n-1} s_i \equiv 0 \equiv n \equiv s_1$. Hence $s(\mathbf{x}) \notin P_n$.

Also solved by Larry E. Askins, J. C. Binz (Switzerland), Ada Booth, David Callan, Con Amore Problem Group (Denmark), Russell Jay Hendel, A. A. Jagers (Netherlands), David W. Koster, John and Libby Krussel, Lamar Problem Solving Group, David E. Manes, Reiner Martin (West Germany), Hans-Peter Nussbaumes (Switzerland), Allan Pedersen (Denmark), The Texas Academy of Mathematics and Science Problem Solving Group, Western Maryland College Problems Group, The Xavier Mathematics Problem League, A. Zulauf (New Zealand), and the proposers.

Morley's analogue for a rhombus **December 1989**

1333. *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*

Prove that the quadrilateral formed by the adjacent quadrisectors of the angles of a rhombus is a square.

Solution by XAMPLE, the XAvier Mathematics Problem LEague, Xavier University, Cincinnati, Ohio.

The angle bisectors of a rhombus are its diagonals, which partition the figure into four congruent right triangles, each of which has its right angle at the center P of the rhombus. So each pair of adjacent quadrisectors of the angles of the rhombus are also the angle bisectors of the acute angles of these four triangles. Suppose they intersect at the points P_1, P_2, P_3, P_4 (labelled clockwise around the quadrilateral, say); these are the vertices of the quadrilateral in which we are interested. Since the three angle bisectors of any triangle are concurrent, each P_i also lies on the angle bisector of the right angle of the triangle in which it lies. But since these four right angles share the vertex P, it follows that the angle bisector PP_1 contains P_3 and that PP_2 contains P_4. Moreover, since the triangles are congruent, the lengths PP_i are all equal. Therefore, $P_1P_2P_3P_4$ is a square.

Also solved by Miguel Amengual (Spain), Francisco Bellot (Spain), Ada Booth, Duane M. Broline, Con Amore Problem Group (Denmark), Timothy Craine and Mark Zager (student), Clayton W. Dodge, John F. Goehl, Jr., Cornelius Groenewoud, Francis M. Henderson, John G. Heuver (Canada), Daniel B. Hirschhorn, Joe Howard and Jon Schlosser and Charles Searcy, Geoffrey A. Kandall, Vaclav Konecny, Lamar Problem Solving Group, Nick Lord (England), Helen M. Marston, Metropolitan MAA Student Chapter in New York City, Robert Patenaude, Amites Sarkar, Volkhard Schindler (East Germany), John Oman, James T. Smith, and the proposer.

Non-adjoint matrix **December 1989**

1334. *Proposed by William P. Wardlaw, U.S. Naval Academy, Annapolis, Maryland.*

Let

$$\mathbf{B} = \begin{pmatrix} 5 & 5 & 2 \\ 5 & 5 & 2 \\ 1 & 1 & 6 \end{pmatrix}.$$

Show that $\mathbf{B}$ is not the classical adjoint of any matrix with real entries.

Solution by Reiner Martin, student, Karlsruhe, West Germany.

Assume that $\mathbf{B}$ is the classical adjoint of $\mathbf{A} = (a_{ij})_{i,j=1,2,3}$. Then $\mathbf{AB} = (\det \mathbf{A}) \cdot \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Thus $5a_{11} + 5a_{12} + a_{13} = 0$ and $2a_{11} + 2a_{12} + 6a_{13} = 0$.

This yields $a_{13}=0$. Similarly we get $a_{23}=0$. Now $\det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}=0$. Therefore, the entry of **B** in the 1st row and 3rd column must be 0, a contradiction.

Also solved by David R. Arterburn, Seung-Jin Bang (Korea), Duane M. Broline, David Callan, Con Amore Problem Group (Denmark), Charlie H. Cooke, Jesse Deutsch, Robert Doucette, Drake University Problem Solving Group, Thomas E. Elsner, Fergus Gaines (Ireland), L. J. Grimm, Daniela Haas (West Germany), G. A. Heuer, A. A. Jagers (Netherlands), Geoffrey A. Kandall, Lamar University Problem Solving Group, Kee-Wai Lau (Hong Kong), Nick Lord (England), David E. Manes, Jean-Marie Monier (France), Marvin Marcus, Larry Olson, Donald H. Pelletier (Canada), Stephen G. Penrice, Harvey Schmidt, Jr., Daniel B. Shapiro, Sahib Singh, John S. Sumner, The Xavier Mathematics Problem League, and the proposer.

Most solutions referred to the known fact that the rank of the classical adjoint of an n-square matrix **A** *is* 0, 1, *or n. (Since the matrix* **B** *has rank* 2, *it cannot be an adjoint.) Shapiro proved a version of the converse: Let* **B** *be an n-square matrix over a field F. Then* **B** *is the classical adjoint of some matrix if either rank* **B** *is* 0, 1, *or n, and in the case of rank n,* $\det \mathbf{B}$ *is an* $(n-1)$*st power of an element of F.*

Divisibility and a three term recurrence **December 1989**

1335. *Proposed by David Callan, University of Bridgeport, Bridgeport, Connecticut.*

Suppose a sequence of integers is defined by $u_0, u_1 \in \mathbf{Z}$, and $u_n = u_{n-1} - (n-1)u_{n-2}$ for $n \geqslant 2$. Show that $n-2$ divides u_n for $n \geqslant 3$.

Solution, without words, by Kevin Ford (student), California State University, Chico, California.

For $n \geqslant 3$,

$$\begin{aligned} u_n &= u_{n-1} - (n-1)u_{n-2} \\ &= (u_{n-2} - (n-2)u_{n-3}) - (n-1)u_{n-2} \\ &= -(n-2)(u_{n-2} + u_{n-3}). \end{aligned}$$

Also solved by Charles Ashbacher, Seung-Jin Bang (Korea), Brian D. Beasley, J. C. Binz (Switzerland), Duane M. Broline, Ada Booth, Con Amore Problem Group (Denmark), Charles R. Diminnie, Francois Dubeau (Canada), Ragnar Dybvik (Norway), Ervin Eltze, Fergus Gaines (Ireland), Jayanthi Ganapathy, Daniela Haas (student, West Germany), Russell Jay Hendel, Joe Howard, Rodney Hutchings (Canada), A. A. Jagers (Netherlands), Geoffrey A. Kendall, Blair Kelly III, John and Libby Krussel, Lamar Problem Solving Group, Graham Lord, Nick Lord (England), David E. Manes, Helen M. Marston, Reiner Martin (West Germany), Jean-Marie Monier (France), Hugh Noland, Laura R. Pasquinelli, Stephen G. Penrice, S. D. Ranasinghe, R. Bruce Richter (Canada), Adam Riese, Volkhard Schindler (East Germany), H.-J. Seiffert (West Germany), Sahib Singh, Aleksandar Slavkovic (student), John S. Sumner, Michael Vowe (Switzerland), T. H. Wang (Canada), Charles H. Webster, Western Maryland College Problem Group, The Xavier Mathematics Problem League, A. Zulauf (New Zealand), and the proposer.

Derangements and determinants **December 1989**

1336. *Proposed by N. J. Lord, Tonbridge School, Kent, England.*

It is known that the number of nonzero terms in the expansion of the $n \times n$ determinant having zeros in the main diagonal and ones elsewhere is equal to d_n, the number of derangements of n objects. (Recall that a derangement is a permutation which 'moves' every object.)

a. Show that the numerical value of the determinant is $(-1)^{n-1}(n-1)$.

b. For which n is it possible to allocate $\pm$ signs to the ones in such a way that d_n is numerically equal to the value of the resulting determinant?

I. Solution to (a) by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Canada, and (b) by Reiner Martin, Karlsruhe, West Germany.

a. Let $D_n = \det(\mathbf{J} - \mathbf{I}_n)$ where J denotes the $n \times n$ matrix with all entries equal to one and $\mathbf{I}_n$ is the $n \times n$ identity matrix. Then adding column j to column one for $j = 2, 3, \ldots, n$ followed by subtracting row one from row i for $i = 2, 3, \ldots, n$ shows that D_n is the determinant of an upper triangular matrix whose main diagonal elements are $n-1, -1, -1, \ldots, -1$. Thus $D_n = (-1)^{n-1}(n-1)$.

b. We will show that the stated property holds if and only if $n \in \{1, 2, 3, 4\}$.

First, we see that

$$\det(0) = 0 = d_1, \qquad \det\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1 = d_2, \qquad \det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 2 = d_3,$$

and

$$\det\begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = 9 = d_4.$$

Now let $n \geqslant 5$. Assume that $\det(a_{ij})_{i,j=1,2,\ldots,n} = d_n$, where $a_{11} = \cdots = a_{nn} = 0$ and $a_{ij} \in \{-1, 1\}$ if $i \neq j$. Then every nonzero term in the expansion of this determinant must have the same sign. This implies

$$a_{14}a_{25} = -a_{15}a_{24},$$
$$a_{13}a_{25} = -a_{15}a_{23},$$
$$a_{13}a_{24} = -a_{14}a_{23}.$$

It is easily checked that this is not possible.

II. Solution by Allan Pedersen, Søborg, Denmark.

a. Let $\mathbf{A}$ be the matrix whose determinant is to be evaluated. It is immediate to verify that $(1, -1, 0, 0, \ldots, 0)^T, (0, 1, -1, 0, \ldots, 0)^T, \ldots, (0, 0, \ldots, 0, 1, -1)^T$ are $n-1$ linearly independent eigenvectors of $\mathbf{A}$ with eigenvalue -1, and that $(1, 1, \ldots, 1)^T$ is an eigenvector with eigenvalue $n-1$. Hence, the characteristic values of $\mathbf{A}$ are -1 with multiplicity $n-1$ and $n-1$ with multiplicity 1, whence $\det(\mathbf{A}) = (-1)^{n-1}(n-1)$.

b. It is possible for $n = 1, 2, 3, 4$, but not for $n > 4$ (see previous solution—Ed.). Let $n > 4$. We will give an estimate for d_n.

$$d_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} = n!\left(e^{-1} - \sum_{j=n+1}^{\infty} \frac{(-1)^j}{j!}\right)$$
$$> n!\left(e^{-1} - \frac{1}{(n+1)!}\right)$$
$$> n!\left(\frac{1}{2.8} - \frac{1}{6!}\right) > \frac{1}{3}n!, \qquad (*)$$

where we have used the alternating series estimate $|\sum_{j=n+1}^{\infty}(-1)^j/j!| < 1/(n+1)!$.

Also, for $n > 4$, let $\mathbf{B}$ be any of the $n \times n$ matrices with zeros in the diagonal and ± 1 on the off-diagonal. By Hadamard's determinant inequality,

$$|\det(\mathbf{B})| \leqslant \left(\sqrt{n-1}\right)^n = (n-1)^{n/2}. \qquad (**)$$

It follows from $(*)$ and $(**)$ and the following lemma that $|\det(\mathbf{B})| < d_n$; hence the impossibility of a matrix $\mathbf{B}$ with the desired property.

LEMMA. $(n-1)^{n/2} < (1/3)n!$ *for* $n > 4$.

Proof. For $n > 4$,

$$\begin{aligned}(n!)^2 &= (1\cdot n)(2\cdot(n-1))(3\cdot(n-2))\cdots((n-1)\cdot 2)(n\cdot 1)\\ &> (n-1)(2(n-1))((9/4)(n-1))(n-1)\cdots(n-1)(2(n-1))(n-1)\\ &= 9(n-1)^n.\end{aligned}$$

We have used the easily verified facts that $x(n+1-x) \geqslant n > n-1$ for $1 \leqslant x \leqslant n$, and that $3(n-2) \geqslant (9/4)(n-1)$ for $n \geqslant 5$.

Also solved by Con Amore Problem Group (Denmark), Joe Howard and Jon Schlosser (part a), A. A. Jagers (The Netherlands), R. Bruce Richter (Canada), Allen J. Schwenk, The Xavier Mathematics Problem League, and the proposer.

Fred Galvin pointed out that part (a) appeared as Problem 6400 in the *American Mathematical Monthly*, October, 1982, p. 602. *Edward T. H. Wang* noted that part (a) is a special case of a couple of problems found in D. K. Faddeev and I. S. Sominskii, *Problems in Higher Algebra*, p. 40. He also pointed out that part (b) is related to a problem due to Pólya involving permanents.

LCM of $[1,2,\dots,n] \geqslant 2^{n-1}$ **December 1989**

1337. *Proposed by G. Behforooz, Utica College of Syracuse University, Utica, New York.*

Prove that the least common multiple of $\{1,2,3,\dots,n\}$ is greater than or equal to 2^{n-1}.

Solution by David W. Koster, University of Wisconsin, La Crosse, Wisconsin.

Let $\lambda(n)$ denote the least common multiple of $\{1,2,3,\dots,n\}$. Note that if n is an odd integer then either $\lambda(n+1) = \lambda(n)$ or $\lambda(n+1) = 2\lambda(n)$ depending on whether $n+1$ fails to be a power of 2 or is a power of 2. In either case we see that if n is odd, then $\lambda(n) \geqslant \frac{1}{2}\lambda(n+1)$. This inequality can clearly be used to establish the desired result $\lambda(n) \geqslant 2^{n-1}$ for an odd positive integer n if it is known to be true for even positive integers. We will thus consider only even positive integers n.

Letting $n = 2m$ we note that $2^{2m} \leqslant 2m\binom{2m}{m}$ and thus

$$2^{2m-1} \leqslant m\binom{2m}{m}. \qquad (*)$$

We will next show that $m\binom{2m}{m}$ is a divisor of $\lambda(2m)$. To this end, for each prime $p \leqslant 2m$ we let $\alpha(p)$ be the unique positive integer satisfying $p^{\alpha(p)} \leqslant 2m < p^{\alpha(p)+1}$ and we note that $\lambda(2m) = \prod_{p \leqslant 2m} p^{\alpha(p)}$.

The exponent of the highest power of a prime p dividing $n!$ is $\sum_{k=1}^{\infty}\lfloor n/p^k \rfloor$ (e.g., see Theorem 6-9, p. 146, in Burton, *Elementary Number Theory*), and from this it follows that the exponent $\beta(p)$ of the highest power of a prime p dividing $\binom{2m}{m}$ is given by $\beta(p) = \sum_{k=1}^{\infty}(\lfloor 2m/p^k \rfloor - 2\lfloor m/p^k \rfloor)$. Since both $\lfloor 2m/p^k \rfloor$ and $\lfloor m/p^k \rfloor$ vanish if $p^k > 2m$ this can be written as

$$\beta(p) = \sum_{k=1}^{\alpha(p)} \left(\lfloor 2m/p^k \rfloor - 2\lfloor m/p^k \rfloor\right). \qquad (**)$$

Since $2[x] \leqslant [2x] \leqslant 2[x] + 1$ for any real number x, each term in the sum above is either 0 or 1 and thus $\beta(p) \leqslant \alpha(p)$. We remark that no prime exceeding $2m$ can be a divisor of $m\binom{2m}{m}$ and we consider a fixed prime $p \leqslant 2m$.

Let $m = p^e q$ where $\gcd(p, q) = 1$. Then in the prime power factorization of $m\binom{2m}{m}$ the exponent of p is $e + \beta(p)$. So to show that $m\binom{2m}{m}$ is a divisor of $\lambda(2m)$ we need only verify that $e + \beta(p) \leqslant \alpha(p)$. If $e = 0$, we have already established this inequality, so we assume that $e \geqslant 1$. Then the first e terms in the sum in $(**)$ will be 0 (since m is divisible by p^k for $1 \leqslant k \leqslant e$) and so even allowing for the maximum contribution (of 1) from each of the remaining $\alpha(p) - e$ terms we have the desired inequality $\beta(p) \leqslant \alpha(p) - e$.

Also solved by Robert A. Agnew, Con Amore Problem Group (Denmark), Kevin Ford (student), Kee-Wai Lau (Hong Kong), Charles H. Webster, and The Xavier Mathematics Problem League.

Comments

Q759. In this problem, proposed by *Norman Schaumberger, Bronx Community College*, a, b, c, and d are the lengths of the sides of a quadrilateral and P is its perimeter. Then

$$abc/d^2 + bcd/a^2 + cda/b^2 + dab/c^2 > P$$

unless $a = b = c = d$.

Murray Klamkin, University of Alberta, makes the following comments.

First, one can obtain the given inequality in one step by an application of Hölder's inequality, i.e.,

$$\begin{aligned}(a^3b^3c^3 + b^3c^3d^3 + c^3d^3a^3 + d^3a^3b^3)^{1/3} &\times (d^3a^3b^3 + a^3b^3c^3 + b^3c^3d^3 + c^3d^3a^3)^{1/3} \\ &\times (c^3d^3a^3 + d^3a^3b^3 + a^3b^3c^3 + b^3c^3d^3)^{1/3} \\ &\geqslant a^3b^2c^2d^2 + b^3c^2d^2a^2 + c^3d^2a^2b^2 + d^3a^2b^2c^2.\end{aligned}$$

For a generalization of this, one can start with the m-th root of the cyclic sum (with respect to the a_i's) whose first term is $a_1^{\alpha_1}a_2^{\alpha_2} \cdots a_n^{\alpha_n}$ and multiplying by $(m-1)$ m-th roots of successive permutations of this sum as above $(1 \leqslant m \leqslant n)$.

The given inequality is also a special case of Muirhead's inequality [Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge University Press, London, 1934, pp. 44–48]:

Let

$$[\alpha_1, \alpha_2, \ldots, \alpha_n] = (1/n!) \sum a_1^{\alpha_1}a_2^{\alpha_2} \cdots a_n^{\alpha_n},$$

where the sum is over the $n!$ terms obtained from $a_1^{\alpha_1}a_2^{\alpha_2} \cdots a_n^{\alpha_n}$ by all possible permutations of the a_i's and $a_i > 0$, $\alpha_i \geqslant 0$. If

$$\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \text{ majorizes } \{\beta_1, \beta_2, \ldots, \beta_n\}, \qquad \beta_i \geqslant 0,$$

that is,

$$\begin{aligned}&\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n, \qquad \beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_n,\\ &\alpha_1 + \alpha_2 + \cdots + \alpha_k \geqslant \beta_1 + \beta_2 + \cdots + \beta_k, \qquad n > k \geqslant 1,\\ &\alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta_1 + \beta_2 + \cdots + \beta_n,\end{aligned}$$

then

$$[\alpha_1, \alpha_2, \ldots, \alpha_n] \succeq [\beta_1, \beta_2, \ldots, \beta_n].$$

Since $\{3,3,3,0\}$ majorizes $\{3,2,2,2\}$, we obtain the original given inequality. It also follows that

$$[9,0,0,0] \succeq [6,3,0,0] \succeq [3,3,3,0] \succeq [3,3,2,1] \succeq [3,2,2,2], \text{ etc.}$$

Answers

Solutions to the Quickies on page 351

A769. Using the weighted AM-GM inequality, we have

$$\sqrt{2} = \left[\sin^2\theta(\csc^2\theta) + \cos^2\theta(\sec^2\theta)\right]^{1/2} \geqslant \left[(\csc^2\theta)^{\sin^2\theta}(\sec^2\theta)^{\cos^2\theta}\right]^{1/2}$$
$$= \frac{1}{(\sin\theta)^{\sin^2\theta}(\cos\theta)^{\cos^2\theta}} = \frac{1}{\sin\theta}(\tan\theta)^{\cos^2\theta} = \frac{1}{\cos\theta}(\tan\theta)^{-\sin^2\theta}.$$

Therefore,

$$2 \geqslant \frac{1}{\sin\theta\cos\theta}(\tan\theta)^{\cos^2\theta - \sin^2\theta}$$

and the result follows.

A770. The constraint condition can be rewritten as

$$1/(1+x) + 1/(1+y) + 1/(1+z) = 1.$$

Then by Jensen's inequality for convex functions,

$$1/(1+x) + 1/(1+y) + 1/(1+z) \geqslant 3/(1+A)$$

where $A = (x+y+z)/3$. Thus, $A \geqslant 2$. Then by the power mean inequality,

$$(x^2+y^2+z^2)/3 \geqslant A^2$$

so that $x^2+y^2+z^2 \geqslant 12$, and with equality if and only if $x = y = z = 2$.

More generally, if $\Sigma 1/(1+x_i) = 1$ and $x_i \geqslant 0$ for $i = 1,2,\dots n$, then $\Sigma x_i^p \geqslant n(n-1)^p$ for $p \geqslant 1$, and with equality if and only if $x_i = n-1$.

A771. We prove the result by induction, the case of $n = 1$ being clear. Assume the result for $n-1$. By the symmetry of the summation, we may assume $x_1 \geqslant x_2 \geqslant \dots \geqslant x_n$. Splitting the right-hand sum in two, depending on whether the subscript n is in the term, yields

$$M = \sum_i m_i - \sum_{i<j} m_{ij} + \sum_{i<j<k} m_{ijk} + \cdots + (-1)^{n+1} m_{12\dots n}$$
$$= \left[\sum_{i\leqslant n-1} m_i - \sum_{i<j\leqslant n-1} m_{ij} + \sum_{i<j<k\leqslant n-1} m_{ijk} + \cdots + (-1)^n m_{12\dots(n-1)}\right]$$
$$+ \left[x_n - \binom{n-1}{1}x_n + \binom{n-1}{2}x_n - \cdots + (-1)^{n+1}x_n\right]$$

The second expression on the right is just $x_n \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j = 0$, so applying the inductive hypothesis completes the proof.

REVIEWS

PAUL J. CAMPBELL, *editor*
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Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematical literature. Readers are invited to suggest items for review to the editors.

Lloyd, Seth, The calculus of intricacy: Can the complexity of a forest be compared to that of *Finnegan's Wake*?, *The Sciences* (September-October 1990), 38-44.

How should one measure complexity of a system? Claude Shannon's "information content," the number of bits it takes to describe the system, is not very satisfactory, for the random typing of a chimpanzee ("each letter is a complete surprise") would rate higher on the information scale than this review. Nor does Gregory Chaitin's "algorithmic information content"—"the size of the shortest algorithm that can generate the number, object, or state under scrutiny"—capture complexity adequately, though it is a good measure of randomness. Charles Bennett's "logical depth" is the number of logical steps that it takes the most plausible algorithm to construct the system. His measure deals summarily with generating a random number X with the one-line program "print X", accommodates repetition well (to print a billion ones: "print 1 a billion times"—not many logical steps there), and rightly attributes depth to "systems" like the 79th Fermat number and the natural outcome of chess (short programs, lots of logical processing), or *Finnegan's Wake*, or a biological system. But Bennett's measure falls across Gödel's undecidability in trying to determine "the most plausible [shortest] algorithm." What Lloyd here proposes in place of these three measures is a measure of the total amount of information processed during the evolution of a system—"identifying the complexity of a thing with the amount of informational and thermodynamic effort involved in putting it together." He proceeds through biological examples to philosophical questions, in an entertaining and thought-provoking essay (even if he takes Gödel undecidability too far). (Note: Lloyd's assertion that "it is hard to see how a number such as .8329513 could be generated by an algorithm shorter than the number itself" is borne out by the absence of this number from Borwein and Borwein's *A Dictionary of Real Numbers*, reviewed elsewhere in this column.)

Bennett, Charles H., Undecidable dynamics, *Nature* (16 August 1990), 606-607.

Exposits the ideas of C. Moore regarding "a simple dynamical representation of undecidibility [sic]." Moore (*Phys. Rev. Lett.* 64, 2354-2357 (1990)) has succeeded in embedding a Turing machine in a continuous dynamical system—in other words, using a dynamical system as a general-purpose ("universal") computer. "Ideally, as Moore points out, one would like to prove computational universality, and undecidability, for some famous problem in continuum dynamics, like the three-body problem." Bennett concludes with the hope that some "exotic chemical brew, say—might have universal computing abilities, just as simpler brews have the ability to crystallize or to generate spontaneous chemical waves and oscillations." (His hope seems appropriate on the brink of Halloween, as I compose this review.)

Peterson, Ivars, Little Fermat: A scheme for speeding up multiplication leads to a unique computer, *Science News* 138 (6 October 1990), 222-223.

While physicists are envisioning universal computers made from exotic materials, mathematicians have put together one that is exotic in its word-size and in its algorithms for arithmetic. "Little Fermat" works with 257-bit words (the limit of current chip technology) and uses arithmetic based on Fermat numbers; it does arithmetic faster than the usual floating-point algorithms while avoiding the errors of rounding. The machine, designed by David V. and Gregory V. Chudnovsky (Columbia) and M. M. Denneau (IBM), was assembled by a single person (Saed G. Younnis). The machine is ideal for number-theoretic calculations (you might have guessed that) but also for digital signal and image processing, as well as for solving differential equations. Says Gregory Chudnovsky, "Even if it remains a one-of-a-kind machine, Little Fermat stands as a demonstration of what should be added to a supercomputer to improve its performance. It would be very cheap to put additional Fermat circuitry into future supercomputers."

Cipra, Barry A., A new wave in applied mathematics: A technique called wavelets may upstage Fourier analysis in a multitude of applications—from CAT scanning to locating subs, *Science* 249 (24 August 1990), 858-859. Wavelet theory sets out the welcome mat, *SIAM News* 23 (5) (September 1990), 8-9.

Wavelet transforms are like Fourier transforms, except that instead of using periodic sines and cosines as the building blocks of other functions, wavelets start from a "mother wavelet" concentrated on a finite interval. They combine it with other wavelets formed by translating it in unit steps and dilating or compressing it by factors of 2. The drawbacks of Fourier transforms—the need to recompute all coefficients if a data error is found, the inability to deal with gaps in the data—do not affect wavelets. This new paradigm has natural uses in data compression and storage (images, speech) and fast numerical algorithms, with specific applications to seismic exploration, improvements in medical imaging, and weather prediction.

Quantum: The Student Magazine of Math and Science. Published by the National Science Teachers Association and Quantum Bureau of the USSR Academy of Sciences, in conjunction with the American Association of Physics Teachers and the National Council of Teachers of Mathematics. Quarterly, 68 pp, annual subscription $18 (individual), $14 (student), $28 (institution/library), $26 (foreign), with bulk subscriptions available. Quantum, 1742 Connecticut Ave. NW, Washington, DC 20009.

Quantum is a new first-class mathematics/physics magazine, designed for high-school students and teachers; but college students and their teachers will enjoy it too. Some articles are translated from the sister publication, the Soviet magazine *Kvant*. The topics are exciting, the writing is excellent, the illustrations are interest-catching, and I've learned something new from every issue. The September/October 1990 issue was the first official one, after two complete pilot issues.

Steen, Lynn Arthur (ed.), *On the Shoulders of Giants: New Approaches to Numeracy*, National Academy Press, 1990; vi + 232 pp, $17.95. ISBN 0-309-04234-8

What mathematics should be learned by today's K-12 students and tomorrow's workforce? This book illustrates five fundamental strands—dimension, quantity, uncertainty, shape, and change—and describes specific teaching approaches with fresh perspectives. The boundaries of mathematics have enlarged beyond simple number and shape to pattern and order of all sorts, and the curriculum of the future—no, the curriculum of the present—must reflect more ways for students to explore patterns. The authors of the five chapters are T. F. Banchoff, J. T. Fey, D. S. Moore, M. Senechal, and I. Stewart, all highly-distinguished expositors.

Borwein, Jonathan, and Peter Borwein, *A Dictionary of Real Numbers*, Wadsworth & Brooks/Cole, 1990; viii + 424 pp, \$73.95. ISBN 0-534-12840-8

"How do we recognize that the number .93371663 ... is actually 2 $\log_{10}(e + \pi)/2$? Gauss observed that the number 1.85407467 ... is (essentially) a rational value of an elliptic integral—an observation that was critical in the development of nineteenth century analysis." This volume contributes part of the answer. It is a "reverse handbook of special function values ... a list of just over 100,000 eight-digit real numbers in the interval [0,1) that arise as the first eight digits of special values of familiar functions.

Browne, Malcolm W., Mathematicians of 3 nations win prestigious Fields Medal, *New York Times* (22 August 1990) (National Edition), A15.

If news of the winners of the Fields Medals made the front page of your paper, I want to hear about it! As I write, two months after the awards, one of the questions on the Newsnet electronic bulletin board's mathematics section is "Who won the Fields Medals?" Since not everyone was in Kyoto last August or was able to find the news in the local paper, I offer a brief note. The winners were Vaughan Jones (Berkeley) [knot theory], Edward Witten (Institute for Advanced Study) [string theory], Vladimir Drinfeld (Institute for Low Temperature Physics and Engineering, Kharkov, USSR) [quantum groups], and Shigefumi Mori (Kyoto University) [algebraic surfaces in three dimensions]. The importance of these Nobel-equivalent prizes, and of mathematics in our culture, can be deduced from the fact that the *New York Times* featured the news on p. 15, just above "Denver Machine Shop Owner Gets Probation in Booby Trap Slaying." Of course, the facts that the Nobel Prizes are worth \$710,000 each (making winning one only slightly less worthwhile—though harder work—than winning your local state lottery), while the Fields Medals carry no monetary award (worth slightly less than winning a free soft drink at your local fast-food restaurant), may have an influence on newsworthiness, not to mention reflecting accurately (and to their satisfaction) most citizens' value of medicine (etc.) over mathematics. Where is the patron mathematics needs, one willing to endow, in perpetuity, Fields Medals prizes to match Nobels? (It would take merely \$25 million, a paltry sum in this age of megabuck art auctions.)

Sterrett, Andrew, *Using Writing to Teach Mathematics*, MAA Notes 16, 1990; xviii + 139 pp, \$15 (P). ISBN 0-88385-066-4

Recent national meetings of the MAA have included overflowing sessions of contributed papers on writing in mathematics classes. The best of those and other papers are included in the 31 essays here, detailing experiments and experiences, and exhibiting a wealth of possibilities—and rewards—for the teacher willing to try some changes. (Congratulations to the designer on an attractive "note"book.)

Ebbinghaus, H.-D., H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neikirch, A. Prestel, and R. Remmert, *Numbers*, Springer-Verlag, 1990; xviii + 391 pp, \$59. ISBN 0-387-97202-1

This translation of the second German edition is of the first volumes in English of a new Springer-Verlag series, Graduate Texts in Mathematics: Readings in Mathematics. One intention of the series is "to make the reader aware that mathematics does not consist of isolated theories, developed side by side, but should be looked upon as an organic whole." To this aim, the authors detail the history and constructions of the number systems, to the complex numbers and the p-adics; provide the classification of real division algebras; and cover nonstandard reals, Conway numbers, and the set theory of infinite cardinal numbers. Despite the series title, the book can be read profitably by undergraduates who have a mentor to get them over hard parts; it makes ideal reading for a senior seminar.

NEWS AND LETTERS

LETTERS TO THE EDITOR

Editor:

In a recent *American Mathematical Monthly* a solution to E3236 [1990, 529-531] used a formula attributed to Hüseyin Demir in an article "Incircles Within" which appeared in this MAGAZINE 59 (1986), 77-83 during your editorship.

Allow me to call your attention to the fact that all the material in Demir's article beginning with Theorem 2 on through the final Theorem 3 is contained in a prior publication, namely, *Amer. Math. Monthly* E1090 [1953, 627] with solution by the proposer [1954, 348-349].

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Editor:

Merrill Barnebey attributes the song whose chorus starts "But if two and two and fifty make a million, ..." to Pete Seeger (this MAGAZINE 63 (1990), 262). Actually, Pete Seeger just composed the music. The words are by Alex Comfort, who, although he is perhaps best known as a cataloguer of sexual techniques, is by profession an expert on geriatrics and has long been active in the peace, racial equality, and other social movements. The song, with the credits, is reprinted from its original appearance in *Sing Out!* in Cooney, Michael, ed., *How Can We Keep From Singing?* (New York: Sing Out, Inc.), 1974, p. 16.

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Editor:

In the October 1990 issue of this MAGAZINE, page 262, the American folksinger Pete Seeger's quote "If two and two and fifty make a million, ..." is followed by a suggestion for "readers ... to try."

Perhaps it is as simple as one-two-three; indeed 1, 2, 3 are exponents in the product $2^1 2^2 50^3$ which equals an exact million.

Prem N. Bajaj
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